Overlapping Generations Model: Equilibrium and Steady State

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1. Steady State and Dynamic Efficiency

Steady State

Definition

A steady state is an equilibrium where all (per capita) variables are constant.

Note: Aggregates can grow $(K_t = k_t N_t)$, but per capita variables cannot (k_t).

The Golden Rule

Definition

The Golden Rule capital stock maximizes steady state consumption (per capita).

Consumption per young household is

$$c^y + c^o/(1+n) = f(k) + (1-\delta)k - (1+n)k'$$

Impose the steady state requirement $k' = k$ and maximize with respect to k:

$$f'(k_{GR}) = n + \delta \tag{1}$$

Intuition...

Dynamic Inefficiency

Definition

An allocation is dynamically efficient, if $k < k_{GR}$.

- $k > k_{GR}$ implies a Pareto inefficient allocation.
- By running down the capital stock, households at all dates could eat more.

Key point:

Nothing rules out a steady state that is dynamically inefficient.

Why is it surprising that the equilibrium can be Pareto inefficient?

Why Is Dynamic Inefficiency Possible?

- Vaguely, the **First Welfare Theorem** says:
  when all markets are competitive and some other conditions hold, every CE is Pareto Optimal.
- One of the "other conditions" comes in 2 flavors:
  1. there is a finite number of goods
  2. $\sum_{j=1}^{\infty} p_j < \infty$ where p_j are the CE (Arrow-Debreu) prices.
- Both conditions are violated in the OLG model.
- Acemoglu, ch. 9.1.

Intuition: Dynamic Inefficiency

- A **missing market**: the old must finance their consumption out of own saving, even if the rate of return is very low.
  - Suppose households value only c^o.
  - Then households save all income at rate of return $f'(k') - \delta$.
  - For high k', this can be negative.
- An alternative arrangement that makes everyone better off:
  - In each period, each young gives up 1 unit of consumption.
  - Each old gets to eat $1+n$ units.
  - If $n > f'(k) - \delta$, this makes everyone better off.
  - Social Security as a potential fix.

2. The Social Planner's Problem

Planner's problem

Imagine an omnipotent **social planner** who

- can assign actions to all agents
  (consumption, hours worked, ...)
- maximizes some average of individual utilities
  "welfare"
- **only faces resource constraints**.

Solving this problem yields **one Pareto optimal** allocation.

- No economy that faces the same technological constraints can do better for everyone.
  - Obvious?
- A benchmark against which equilibria can be assessed.
- But there may be many Pareto optimal allocations.

2.1. OLG Welfare function

The planner maximizes a weighted average of individual utilities.

Welfare is

$$\underbrace{\mu_0 \beta u(c_1^o)}_{\text{initial old}} + \sum_{t=1}^{\infty} \underbrace{\mu_t [u(c_t^y) + \beta u(c_{t+1}^o)]}_{\text{generation } t}$$

Old consumption of the initial old is the earliest quantity that the planner can change.

Planner Constraints

The planner only faces feasibility or **resource constraints**.

In this model:

$$Y = C + I \tag{2}$$

$$\underbrace{F(K_t, N_t)}_{Y} = \underbrace{N_t c_t^y + N_{t-1} c_t^o}_{C} + \underbrace{K_{t+1} - (1-\delta) K_t}_{I} \tag{3}$$

Or, in per capita young terms $(k_t = K_t / N_t)$:

$$f(k_t) = c_t^y + c_t^o/(1+n) + (1+n)k_{t+1} - (1-\delta)k_t$$

because $K_{t+1}/N_t = (K_{t+1}/N_{t+1}) \times (N_{t+1}/N_t)$

Planner's Lagrangian

$$
\begin{aligned}
\Gamma \;=\;& \mu_0 \beta u(c_1^o) + \textstyle\sum_{t=1}^{\infty} \mu_t [u(c_t^y) + \beta u(c_{t+1}^o)] \\
& + \textstyle\sum_{t=1}^{\infty} \lambda_t \left[\begin{array}{c} (1-\delta)k_t + f(k_t) \\ -c_t^y - c_t^o/(1+n) - (1+n)k_{t+1} \end{array} \right]
\end{aligned}
$$

Planner's FOCs:

$$
\begin{aligned}
\mu_t u'(c_t^y) &= \lambda_t \\
\mu_{t-1} \beta u'(c_t^o) &= \lambda_t/(1+n) \\
\lambda_{t+1}[1-\delta+f'(k_{t+1})] &= \lambda_t(1+n)
\end{aligned}
$$

Interpretation

Three ways of using a unit of goods at date t:

$$\lambda_t = \mu_t u'\left(c_t^y\right) \tag{4}$$

$$\lambda_t = (1+n)\,\mu_{t-1} u'\left(c_t^o\right) \tag{5}$$

$$\lambda_t = \frac{f'\left(k_{t+1}\right) + 1 - \delta}{1+n} \lambda_{t+1} \tag{6}$$

All uses must give the same marginal utility (λ_t).

Planner's problem

Static optimality:

$$\lambda_t = \mu_t u'(c_t^y) = \mu_{t-1}(1+n)\beta u'(c_t^o)$$

Intuition...

Euler equation

$$\mu_t u'(c_t^y)[1-\delta+f'(k_t)] = \mu_{t-1} u'(c_{t-1}^y)(1+n)$$

Using the static condition, the Euler equation becomes

$$u'(c_t^y) = \beta u'(c_{t+1}^o)[1-\delta+f'(k_{t+1})] \tag{7}$$

which looks like the Euler equation of the household.

This is not surprising: the planner should respect the individual FOCs unless there are externalities.

Interpretation of the Euler equation

- A feasible perturbation does not change welfare.
- In $t-1$:
  - $c_{t-1}^{y} \downarrow$ by $(1+n)$
  - $k_t \uparrow$ by 1 (per capita of the date t young)
- In t:
  - output $\uparrow$ by $f'(k_t)$ (per capita t young)
  - raise c_t^y by $1-\delta+f'(k_t)$ or
  - raise c_t^o by $(1+n)(1-\delta+f'(k_t))$
- From $t+1$ onwards: nothing changes
  - especially not k_{t+1}

Planner's Solution

Sequences $\{c_t^y, c_t^o, k_{t+1}\}_{t=1}^{\infty}$ that satisfy:

- Static and Euler equation.
- Feasibility.
- A transversality condition or $k_{t+1} \geq 0$.
  - We talk about those later.

2.2. Comparison with Equilibrium

The same:

- Euler equation
- Resource constraint = goods market clearing.

Different:

- CE has 2 budget constraints (one redundant by Walras' law)
- Planner has static condition

Missing in the C.E.: a mechanism for transferring goods from young to old (planner's static condition).

Planner's Steady State

Euler in steady state:

$$\frac{\mu_t}{\mu_{t-1}} u'(c^y)[1-\delta+f'(k)] = u'(c^y)(1+n)$$

For a steady state to exist, weights must be of the form

$$\mu_t = \omega^t, \quad \omega < 1$$

Otherwise the ratios μ_{t+1}/μ_t in the FOCs are not constant.

Then the Euler equation becomes

$$\omega\ (1-\delta+f'(k_{MGR})) = (1+n)$$

This is the **Modified Golden Rule**. ($\omega = 1$ is the Golden Rule).

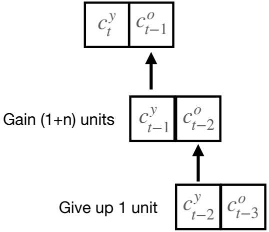
Because $\omega < 1$: $k_{MGR} < k_{GR}$ and the MGR is **dynamically efficient**.

How does the planner avoid dynamic inefficiency?

If the planner desires lots of old age consumption, he can implement a "transfer scheme" of the following kind:

Take a unit of consumption from each young and give $(1+n)$ units to each old at the same date.

There is no need to save more than the GR.



Of course, there aren't really any transfers in the planner's world.

3. Final Example: Government Bonds

Final Example: Government Bonds

We introduce harmless bonds into the model.

All the government does: issue new bonds to pay off the old ones.

Magical result: the steady state is at the golden rule.

One insight: **introducing an infinitely lived asset fixes dynamic inefficiency**

- actually, the assets here live for only one period
- but they serve the same function because there is now an infinitely lived agent who keeps trading the bonds

Environment

Demographics: $N_t = (1+n)^t$. Agents live for 2 periods.

Preferences:

$$(1-\beta)\ln(c_t^y) + \beta\ln(c_{t+1}^o)$$

Endowments:

- The initial old are endowed with s_0 units of capital.
- Each young is endowed with one unit of work time.

Environment

Technology:

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

Government: The government only rolls over debt from one period to the next:

$$B_{t+1} = R_t B_t$$

Markets: for goods, bonds, labor, capital rental.

Household Solution

The household solves

$$\max(1-\beta)\ln(w-s)+\beta\ln(R's) \tag{8}$$

The FOC is

$$c'/c = R'\beta/(1-\beta) \tag{9}$$

Therefore

$$s = (w-s)\beta/(1-\beta) \tag{10}$$

and thus

$$s = \beta w \tag{11}$$

Firm Solution

This is standard:

$$
\begin{aligned}
r &= f'(k) = \alpha k^{\alpha-1} \\
w &= f(k) - f'(k)k = (1-\alpha)k^{\alpha}
\end{aligned}
$$

where $k = K/L$.

Questions

1. Define a competitive equilibrium.
2. Derive the law of motion for the capital stock

$$k_{t+1}(1+n) = \beta(1-\alpha)k_t^{\alpha} - b_{t+1}(1+n) \tag{12}$$

   where $b = B/L$.
3. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ?
4. Derive the steady state capital stock for $b > 0$.
5. Show that the capital stock is lower in the steady state with positive debt (crowding out).

Reading

- Acemoglu (2009), ch. 9.
- Krueger, "Macroeconomic Theory," ch. 8
- Ljungqvist and Sargent (2004), ch. 9 (without the monetary parts).
- McCandless and Wallace (1991)and De La Croix and Michel (2002) are book-length treatments of overlapping generations models.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

De La Croix, D. and P. Michel (2002): *A theory of economic growth: dynamics and policy in overlapping generations*, Cambridge University Press.

Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.

McCandless, G. T. and N. Wallace (1991): *Introduction to dynamic macroeconomic theory: an overlapping generations approach*, Harvard University Press.