Cash-in-Advance Model

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1. Introduction

We study a second model of money.

Models where money is a bubble (such as the OLG model we studied) have 2 shortcomings:

- They fail to explain rate of return dominance.
- Money has no transaction value.

CIA models focus on **transactions demand** for money.

The basic idea:

- There is a **technology** that produces liquidity services.
- Buying certain goods requires those services.
- The simplest case: only cash can be used to buy those goods.

2. Environment

Demographics:

- a representative household of mass 1
- no firms; households operate the technology

Preferences: $\sum_{t=1}^{\infty} \beta^t u(c_t)$

Endowments at $t = 1$:

- m_{t-1}^d units of money;
- k_1 units of the good

Technologies:

- $f(k_t) + (1-\delta)k_t = c_t + k_{t+1}$

Environment

Government

- Costlessly prints τ_t units of money and hands it to households (lump-sum)
- The nominal money growth rule is

$$\tau_t = g \times M_{t-1}$$

  $g > 0$ is a parameter

Markets:

- goods: price p_t
- money: price 1

Notation:

- m is what the household chooses
- M is aggregate money
- in equilibrium, they are the same

Cash-in-advance constraint

Transactions technology:

$$m_t/p_t \geq c_t + k_{t+1} - (1-\delta)k_t \tag{1}$$

Requires that **some goods are purchased with money**.

An odd feature:

- the CIA constraint really is a technology
- its output: transactions services
- its input can be produced at no cost

Money

At the start of t:

- households hold m_{t-1}^d
- the government hands out τ_t (lump sum)

Households now hold $m_t = m_{t-1}^d + \tau_t$

- This can be used to buy things in t
- But no money changes hands (just have to have in your pocket).

Purchased money is taken into $t+1$: m_t^d

Note that money earned in period t cannot be used until $t+1$.

3. Household

Budget constraint:

$$k_{t+1} + c_t + m_t^d / p_t = f(k_t) + (1-\delta) k_t + m_t / p_t$$

Savings are taken into the next period in the form of capital and money

Household problem

We simply add one constraint to the household problem: the CIA constraint.

The household solves

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t)$$

subject to the budget constraint

$$k_{t+1} + c_t + m_t^d / p_t = f(k_t) + (1-\delta)k_t + m_t / p_t$$

and the CIA constraint

$$m_t / p_t \geq c_t + k_{t+1} - (1-\delta)k_t$$

and the law of motion

$$m_{t+1} = m_t^d + \tau_{t+1}$$

Household problem

Remarks

- Exactly what kinds of goods have to be bought with cash is arbitrary.
- The CIA constraint holds with equality if the rate of return on money is less than that on capital (the nominal interest rate is positive).

3.1. Household: Dynamic Program

Individual state variables: m, k.

Bellman equation:

$$\begin{aligned} V(m,k) &= \max u(c) + \beta V(m',k') \\ &\quad + \lambda(BC) + \gamma(CIA) \end{aligned}$$

We need to impose

$$m_t = m_{t-1}^d + \tau_t$$

Then we can use m_{t+1} as a control (this would not work under uncertainty).

Bellman Equation

$$
\begin{aligned}
V(m,k) &= \max u(c) + \beta V(m',k') \\
&\quad + \lambda[f(k) + (1-\delta)k + m/p - c - k' - (m' - \tau')/p] \\
&\quad + \gamma[m/p - c - k' + (1-\delta)k]
\end{aligned}
$$

$\lambda > 0$: multiplier on budget constraint

γ : multiplier on CIA constraint - could be 0.

Notes:

- We could substitute the budget constraint into $u(c)$, but it's instructive not to.
- We cannot sub out the CIA constraint (why not?)

First-order conditions

$$
\begin{aligned}
u'(c) &= \lambda + \gamma \\
\beta V_m(\bullet') &= \lambda / p \\
\beta V_k(\bullet') &= \lambda + \gamma
\end{aligned}
$$

Envelope conditions:

$$
\begin{aligned}
V_m &= (\lambda + \gamma)/p \\
V_k &= \lambda[f'(k) + 1 - \delta] + \gamma[1 - \delta]
\end{aligned}
$$

Interpretation ...

Simplify

Simplify (eliminate V's and $\lambda+\gamma$'s):

$$
\begin{aligned}
u'(c)/\beta &= \lambda' f'(k') + [1-\delta]u'(c') \\
\beta u'(c')p/p' &= \lambda \\
u'(c) &= \lambda + \gamma
\end{aligned}
$$

Interpretation...

Kuhn Tucker:

$$
\begin{aligned}
\gamma[m/p - c - k' + (1-\delta)k] &= 0 \\
\gamma &\geq 0
\end{aligned}
$$

Household: Solution

A solution to the household problem: $\{c_t, m_{t+1}, k_{t+1}, \lambda_t, \gamma_t\}$ that solve

1. 3 FOCs
2. budget constraint
3. either CIA constraint or $\gamma = 0$
4. transversality conditions

$$\lim_{t\to\infty} \beta^t \, u'(c_t) \, (k_t + m_t/p_t) = 0 \tag{2}$$

Note: There is one TVC for the total value of assets.

- to see why, think about red and green capital...

3.2. Household: CIA does not bind

With $\gamma = 0$:

$$\begin{aligned} \beta \lambda' / p' &= \lambda / p \\ \lambda / \beta &= \lambda' [f'(k') + 1 - \delta] \\ u'(c) &= \lambda \end{aligned}$$

Standard Euler equation:

$$u'(c) = \beta u'(c') \left[f'(k') + 1 - \delta \right] \tag{3}$$

"No arbitrage" condition:

$$f'(k') + 1 - \delta = p / p' \tag{4}$$

Interpretation...

When does the CIA constraint bind?

No arbitrage:

$$1+i=(1+r)(1+\pi)=[f'(k)+1-\delta]\;p'/p=1$$

The CIA constraint binds unless the return on money equals that on capital

- i.e. the nominal interest rate is zero.

Holding money has no opportunity cost.

The presence of money does not distort the intertemporal allocation.

We have the standard Euler equation.

Household solution

Sequences $\{c_t, m_t, k_t\}$

that satisfy

1. Euler equation
2. budget constraint
3. no arbitrage

Plus boundary conditions

3.3. Binding CIA constraint

Euler equation:

$$u'(c) = \beta^2 u'(c'')(p'/p'')f'(k') + (1-\delta)\beta u'(c') \tag{5}$$

Today:

- Give up $dc = -\varepsilon$.

Tomorrow:

- $dk' = \varepsilon$.
- Eat the undepreciated capital: $dc' = (1-\delta)\varepsilon$.
- Produce additional output $f'(k')\varepsilon$.
- Save it as money: $dm'' = f'(k')\varepsilon\, p'$.

The day after:

- Eat an additional dm''/p''.

Household Problem

Why isn't there a simple Euler equation for the perturbation:

1. $dc = -\varepsilon$. $dm' = p\varepsilon$.
2. $dc' = \varepsilon \; p/p'$.

The Euler equation for this perturbation is:

$$\begin{aligned} u'(c) &= \lambda + \gamma \\ &= \beta u'(c') \; p/p' + \gamma \end{aligned}$$

Household Solution

Sequences $\{c_t, m_t, k_t\}$

that satisfy:

1. Euler equation
2. budget constraint
3. CIA constraint

Plus boundary conditions

4. Equilibrium

Government

The government's only role is to hand out lump-sum transfers of money.

The nominal money growth rule is

$$\tau_t = g \times M_{t-1}$$

$g > 0$ is a parameter

Money holdings in period t are

$$\begin{aligned} M_t &= M_{t-1} + \tau_t \\ &= (1+g)M_{t-1} \end{aligned}$$

Market clearing

- Goods: $c + k' = f(k) + (1 - \delta)k$.
- Money market: $M_t = m_t$

Equilibrium

An **equilibrium** is a sequence

that satisfies

5. Steady State

Steady state properties

Objects: $c, k, m/p, \pi$

Constant m/p requires

$$\frac{m_{t+1}}{m_t} = 1 + g = \frac{p_{t+1}}{p_t} = 1 + \pi \tag{6}$$

Steady State: CIA does not bind

$$
\begin{aligned}
f'(k) + 1 - \delta &= (1+g)^{-1} \text{ (no arbitrage)} && (7) \\
&= 1/\beta \text{ (Euler)} && (8) \\
f(k) - \delta k &= c \text{ (R.C.)} && (9)
\end{aligned}
$$

Result:

A steady state where CIA does not bind only exists if $\beta = 1 + g$.

Intuition...

Then: The steady state coincides with the (Pareto optimal) non-monetary economy.

Binding CIA constraint

The Euler equation implies

$$1 = \beta^2 (1+\pi)^{-1} f'(k') + (1-\delta)\beta$$

Using $1+\pi = 1+g$ this can be solved for the capital stock:

$$f'(k_{ss}) = (1+g)[1-\beta(1-\delta)]/\beta^2 \tag{10}$$

When $\beta = 1+g$, this coincides with k_{ss} when the CIA constraint does not bind.

When does CIA constraint bind?

Steady state return on money: $(1+g)^{-1}$

If $(1+g) = \beta$:

- return on money equals return on capital (equals discount factor)
- CIA does not bind

Higher g reduces k_{ss} and increases return on capital

Therefore: CIA binds when $(1+g) > \beta$

Properties: Binding CIA

CIA implies:

$$f(k) = m/p \tag{11}$$

Goods market clearing with constant k implies

$$c = f(k) - \delta k \tag{12}$$

A steady state is a vector $(c, k, m/p)$ that satisfies (10) through (12).

Properties: Binding CIA

Definition

Money is called **neutral** if changing the level of M does not affect the real allocation. It is called **super neutral** if changing the growth rate of M does not affect the real allocation.

Money is not super neutral

- Higher inflation (g) implies a lower k.
- Inflation increases the cost of holding money, which is required for investment (inflation tax).

Properties: Binding CIA

Exercise:

- Show that super-neutrality would be restored, if the CIA constraint applied only to consumption $(m/p \geq c)$.
- Intuition: investment is no longer subject to the inflation tax.

Properties: Binding CIA

The velocity of money is one

- Higher inflation reduces money demand only be reducing output.
- This is a direct consequence of the rigid CIA constraint and probably an undesirable result.
- Obviously, this would not be a good model of hyperinflation.
- This limitation can be avoided by changing the transactions technology (see RQ).

What if $(1+g) < \beta$

There is no steady state with $1+g < \beta$

The reason:

- money would offer a rate of return above the discount rate
- the household would choose unbounded consumption.
- Cf. the Euler equation
$$u'(c) = \beta R \, u'(c') \tag{13}$$
with $R = (1+g)^{-1}$ for holding money.

What would the equilibrium look like?

Optimal Monetary Policy

- **The Friedman rule maximizes steady state welfare.**
- Friedman Rule: Set nominal interest rate to 0.
- Proof: Under the Friedman rule, the steady state conditions of the CE coincides with the non-monetary economy's.
- Intuition:
  - It is optimal to make holding money costless b/c money can be costlessly produced.
  - This requires that the rate of return on money $\frac{1}{1+\pi}$ equal that on capital.

Is this a good theory of money?

Recall the central questions of monetary theory:

1. Why do people hold money, an asset that does not pay interest (rate of return dominance)?
2. Why is money valued in equilibrium?
3. What are the effects of monetary policy: one time increases in the money supply or changes in the money growth rate?

Is this a good theory of money?

Positive features:

1. Rate of return dominance.
2. Money plays a liquidity role.

Drawbacks:

1. The reason why money is needed for transactions is not modeled.
2. The form of the CIA constraint is arbitrary (and important for the results).
3. The velocity of money is fixed.

Reading

- Blanchard & Fischer (1989), 4.2.