Derivation of the Hamilton–Jacobi–Bellman Equation with Poisson Shocks

Setup

Consider an asset whose value is the expected discounted present value of a flow profit stream:

$$V = E \int_0^\infty e^{-rt}\, \pi(t)\, dt. \tag{1}$$

Profits evolve deterministically except for jumps governed by a Poisson process with flow probability ρ. When the shock arrives, the continuation value jumps to $\hat{V}$. We derive the associated HJB equation:

$$rV = \pi + \dot{V} + \rho\left(\hat{V} - V\right). \tag{2}$$

Step 1: Split the integral over a small interval

Break the integral at a short interval dt. Over $[0, dt]$, approximate $e^{-rt} \approx 1$ and $\pi(t) \approx \pi$, so the flow payoff over the interval is $\pi\, dt$. The remainder gives the continuation value at dt, discounted back:

$$V = \pi\, dt + e^{-r\, dt}\, E\big[V(t + dt)\big]. \tag{1}$$

Step 2: Expand the discount factor

For small dt, use the first-order Taylor expansion $e^{-r\, dt} \approx 1 - r\, dt$:

$$V = \pi\, dt + (1 - r\, dt)\, E\big[V(t + dt)\big]. \tag{2}$$

Step 3: Evaluate the expected continuation value

During the interval dt, two things can happen:

- With probability $\rho\, dt$: the Poisson shock arrives and the continuation value becomes $\hat{V}$.
- With probability $1 - \rho\, dt$: no shock, and the value evolves smoothly to $V + \dot{V}\, dt$.

Therefore:

$$E\big[V(t + dt)\big] = (1 - \rho\, dt)\big(V + \dot{V}\, dt\big) + \rho\, dt\, \hat{V}. \tag{3}$$

Step 4: Substitute and expand

Plug the expected continuation value into the Bellman equation:

$$V = \pi\, dt + (1 - r\, dt)\Big[(1 - \rho\, dt)\big(V + \dot{V}\, dt\big) + \rho\, dt\, \hat{V}\Big]. \tag{4}$$

Expand the right-hand side, keeping only terms up to first order in dt. All terms involving $(dt)^2$ or higher vanish in the limit:

$$V = \pi\, dt + V + \dot{V}\, dt - \rho\, dt\, V + \rho\, dt\, \hat{V} - r\, dt\, V. \tag{5}$$

Step 5: Cancel and collect terms

Subtract V from both sides:

$$0 = \pi\, dt + \dot{V}\, dt - \rho\, dt\, V + \rho\, dt\, \hat{V} - r\, dt\, V. \tag{6}$$

Divide through by dt:

$$0 = \pi + \dot{V} + \rho\, \hat{V} - \rho\, V - rV. \tag{7}$$

Rearrange to obtain the HJB equation:

$$\boxed{rV = \pi + \dot{V} + \rho\Big(\hat{V} - V\Big)} \tag{8}$$

Interpretation

The left-hand side, rV, is the required return on the asset—what an investor would earn on a risk-free asset of equal value. The right-hand side decomposes into three components:

- π — the flow payoff (dividends).
- $\dot{V}$ — the expected capital gain from smooth, deterministic changes in value.
- $\rho(\hat{V} - V)$ — the expected capital gain (or loss) from the Poisson event: the jump size $(\hat{V} - V)$ weighted by its flow probability ρ.

This is the continuous-time analogue of the standard Bellman equation in discrete time. The no-arbitrage condition requires that the return on holding the asset equals the total expected gain from all sources.