

Stochastic Dynamic Programming: Theorems

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Econ720

November 15, 2021

Theorems: Stochastic DP

We state conditions under which dynamic programming “works”
The assumptions needed and the results are very similar to the deterministic case.

Acemoglu (2009) has a simplified version with discrete random variables.

Stokey et al. (1989) have more general results.

The generic problem

Environment

Start with capital stock $x(0)$.

The shocks $z(t)$ follow a discrete Markov chain.

- ▶ A strong assumption.
- ▶ It can be relaxed without affecting results too much.
- ▶ But at the expense of notation.

The generic problem

After each node $z^t = (z(0), \dots, z(t))$ of shocks, choose next period's capital stock

$$x(t+1) = x[z^t] \quad (1)$$

$x(t+1)$ is constrained to lie in the set $G(x(t), z(t))$.

Key: period utility and constraints only depend on current realizations $(x(t), z(t))$, not on history.

If not: DP fails (or we need tricks).

P1: Sequence problem

$$V^*(x(0), z(0)) = \max_{\{x[z^t]\}} E_0 \sum_{t=0}^{\infty} \beta^t U(x[z^{t-1}], x[z^t], z(t))$$

subject to

$$x[z^t] \in G(x[z^{t-1}], z(t))$$

$x(0)$ given

The law of motion for x is built into U .

P2: Recursive problem

$$V(x, z) = \max_{y \in G(x, z)} U(x, y, z) + \beta E[V(y, z') | z]$$

So much simpler!

Assumptions

A1:

▶ $G(x, z)$ is nonempty.

▶ For all feasible plans:

$$\lim_{n \rightarrow \infty} E \left[\sum_{t=0}^{\infty} \beta^t U(x[z^{t-1}], x[z^t], z(t)) \mid z(0) \right] < \infty.$$

A **feasible plan** is now a collection of state contingent plans ($x[z^t]$ for all histories z^t) that satisfies $x[z^t] \in G$.

- ▶ X is a compact subset of \mathbb{R}^K
 - ▶ where $x(t) \in X$.
- ▶ U is continuous.

[There are additional issues when z does not live in a finite set]

Theorem 1: Equivalence of values

- ▶ Assume A1 and A2.
- ▶ $V^*(x, z)$ in the sequence problem solves the recursive problem.
- ▶ $V(x, z)$ in the recursive problem equals $V^*(x, z)$ in the sequence problem, IF

$$\lim_{t \rightarrow \infty} \beta^t E [V(x [z^{t-1}], z(t))] = 0$$

for all feasible plans.

Theorem 2: Principle of Optimality

- ▶ Assume A1 and A2.
- ▶ Any optimal plan in the sequence problem satisfies the Bellman equation with value V^* :

$$V^*(x[z^{t-1}], z(t)) = U(x[z^{t-1}], x[z^t], z(t)) + \beta E[V^*(x[z^t], z(t+1)) | z(t)]$$

- ▶ Any feasible plan that solves the above attains V^* in the sequence problem.

The point: Solving P1 and P2 are equivalent.

Theorem 3: Existence of solutions

- ▶ Assume A1 and A2.
- ▶ An optimal plan exists for any initial conditions $x(0), z(0)$.
- ▶ V is unique, continuous, bounded in x for each z .

- ▶ $U(x, y, z)$ is strictly concave in the sense that

$$U(\bar{x}, \bar{y}, z) \geq \alpha U(x, y, z) + (1 - \alpha) U(x', y', z) \quad (2)$$

with strict inequality when $x \neq x'$, where $\bar{x} = \alpha x + (1 - \alpha)x'$ and $\bar{y} = \alpha y + (1 - \alpha)y'$.

- ▶ The set $G(x, z)$ is convex in the sense that

$$y \in G(x), y' \in G(x') \implies \bar{y} \in G(\bar{x}) \quad (3)$$

- ▶ $U(x,y,z)$ is strictly increasing in all elements of x .
- ▶ G is monotone: $x \leq x' \implies G(x,z) \subset G(x',z)$ for all z .

A5

- ▶ U is continuously differentiable in x .

Theorem 4: Concavity of V

- ▶ Assume A1, A2, A3.
- ▶ Then V is strictly concave in x .
- ▶ The optimal plan $x[z^t] = \pi(x(t), z(t))$ is unique and π is continuous in x .

Recall A3: U strictly concave. G convex.

Then the Bellman equation is a concave optimization problem.

Theorem 5: Monotonicity of V

- ▶ Assume A1, A2, A4.
- ▶ Then V is strictly increasing in x .

Recall A4: U and G are monotone in x .

Theorem 6: Differentiability of V

- ▶ Assume A1-A3, A5.
- ▶ Then V is continuously differentiable in x .

Recall A5: U is differentiable.

Theorem 8: Euler equations

- ▶ Assume A1-A5.
- ▶ Then the interior feasible plan that satisfies the Euler equation

$$D_y U(x, \pi(x, z), z) + \beta E [D_x U(\pi(x, z), \pi(\pi(x, z), z'), z') | z] = 0$$

and the TVC

$$\lim_{t \rightarrow \infty} \beta^t E [D_x U(t) \pi(t)] = 0$$

solves the recursive problem P2.

- ▶ The Euler equation with scalar x :

$$\frac{\partial U(x, x', z)}{\partial x'} + \beta E \left[\frac{\partial U(x', x'', z')}{\partial x'} \Big| z \right] = 0 \quad (4)$$

- ▶ Note: the TVC must hold starting from any node $x[z^t], z(t)$.

Continuous shocks

- ▶ If the shock z lives in a continuum, nothing of substance changes.
- ▶ Acemoglu 16.4

Example: Permanent income hypothesis

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- ▶ A household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c(t)) \quad (5)$$

- ▶ u has all nice properties: strictly increasing, concave, differentiable.
- ▶ Budget constraint:

$$a(t+1) = (1+r)a(t) + w(t) - c(t) \quad (6)$$

- ▶ $w(t)$ is i.i.d. with $\Pr(w(t) = w_j) = q_j$.

Budget constraint

- ▶ A tricky issue: the budget constraint.
- ▶ How much should the household be able to borrow?
- ▶ A natural borrowing constraint:

$$a(t) \geq - \sum_{s=0}^{\infty} \frac{w_1}{(1+r)^s} \equiv -b_1 \quad (7)$$

- ▶ This ensures that the household can repay his debts, even if he receives the worst possible income realization in each period.
- ▶ Because w is i.i.d. and the household lives forever, b_1 is a constant.

Sequence problem

- ▶ The history is w^t .
- ▶ The household chooses $c[w^t]$ for all possible histories.
- ▶ The problem is too tedious to even write down...

Recursive formulation

$$V(a, w) = \max_{a' \in [-b_1, (1+r)a+w]} u([1+r]a + w - a') + \beta EV(a', w') \quad (8)$$

Mapping into the generic problem:

- ▶ $x \rightarrow a, z \rightarrow w$
- ▶ $G(z, x) \rightarrow [-b_1, (1+r)a + w]$
- ▶ $U(x, y, z) \rightarrow u([1+r]a + w - a')$

First-order conditions

Verify A1-A5 ...

Then we can characterize the solution by the FOCs:

$$u'(c) = \beta EV_a(a', w') \quad (9)$$

$$V_a(a, w) = (1+r)u'(c) \quad (10)$$

Euler:

$$u'(c) = \beta (1+r)Eu'(c') \quad (11)$$

Verify A1

- ▶ $G(x, z) = [-b_1, (1+r)a + w]$ is nonempty – b_1 is constructed that way.
- ▶ For all feasible plans:
$$\lim_{t \rightarrow \infty} E \left[\sum_{t=0}^{\infty} \beta^t U(x[z^{t-1}], x[z^t], z(t)) \mid z(0) \right] < \infty.$$
 - ▶ This is NOT generally satisfied.
 - ▶ $(1+r) > \beta$ could imply unbounded growth.
 - ▶ We need a restriction that r not too high. Tedious details...

Verify A2

- ▶ X is a compact subset of \mathbb{R}^K
 - ▶ Here: $X = \mathbb{R}_+$ which is obviously not compact.
 - ▶ We need to argue that bounding a from above does not bind (when $1 + r < \beta$).
- ▶ U is continuous – by assumption.

Verify A3

- ▶ $U(x, y, z)$ is strictly concave in the sense that

$$U(\bar{x}, \bar{y}, z) \geq \alpha U(x, y, z) + (1 - \alpha) U(x', y', z) \quad (12)$$

with strict inequality when $x \neq x'$, where $\bar{x} = \alpha x + (1 - \alpha)x'$ and $\bar{y} = \alpha y + (1 - \alpha)y'$.

- ▶ Here: $U([1+r][\alpha a_1 + (1-\alpha)a_2] + w - [\alpha a'_1 + (1-\alpha)a'_2])$
with $\partial U / \partial a' < 0$ and $\partial^2 U / \partial (a')^2 < 0$.
- ▶ The set $G(x, z)$ is convex in the sense that

$$y \in G(x), y' \in G(x') \implies \bar{y} \in G(\bar{x}) \quad (13)$$

- ▶ easy to check

Verify A4

- ▶ $U(x,y,z)$ is strictly increasing in all elements of x .
 - ▶ Here: $\partial u / \partial a > 0$.
- ▶ G is monotone: $x \leq x' \implies G(x,z) \subset G(x',z)$ for all z .
 - ▶ Here: $(1+r)a$ is increasing in a .

Verify A5

- ▶ U is continuously differentiable in x . – by assumption.

Quadratic case

▶ Assume $u(c) = \phi c - 0.5c^2$

▶ $u'(c) = \phi - c$.

▶ Euler:

$$\phi - c = \beta(1+r)E[\phi - c'] \quad (14)$$

▶ Nothing in the info set at t should predict consumption growth [in this example: $Ec' - c$]. Hall 1978.

▶ Strangely, a large literature has tested this prediction, even though it only holds with quadratic utility!

Shortcut

- ▶ We could have derived the Euler equation naively by treating E as a constant in the optimization problem.
- ▶ The deterministic FOCs turn out to be correct (in many [all?] cases).

Reading

- ▶ Acemoglu (2009), ch. 16.1-16.2.
- ▶ Krusell (2014), ch. 6.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .