Asset Pricing: Extensions

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State contingent claims

- Some assets pay out only in particular states of the world
  - e.g. insurance contracts.
- Standard asset pricing formulas apply to those assets.
- It just adds notation...

State contingent claims

- We start from the Lucas fruit tree model.
- In addition to stocks and bonds, households can purchase assets that pay out in exactly one state of the world.
  - Arrow securities or state contingent claims
- Their role in theory:
  - given a sufficiently rich set of Arrow securities, we can replicate any asset
  - can set up a model with all possible insurance opportunities (complete markets)

Notation

- quantity purchased of asset that pays out in state d': $y'(d'|d)$.
  - for convenience just write $y'(d')$
- price of that asset: $q(d'|d)$.

Household

States: all assets held, $k, b,$ and all $y(d)$.

- call that s

Choices: $b', k', y'(d')$ for all d'.

Dynamic Program:

$$V(s,d) = \max_{c,k',b',y'(d')} u(c) + \beta EV(s', d')$$

subject to

$$Rb + (p+d)k + y(d) = c + b' + pk' + \sum_{d'} q(d'|d)\, y'(d')$$

Note: only the y matching the realized value of d pays out.

First-order conditions for state contingent claims

Applying the Lucas asset pricing equation:

$$1 = \mathbb{E}\left\{ MRS_{t+1} \frac{\mathbb{I}(d')}{q(d',d)} \right\}$$

Special feature of Arrow securities: Only one term in the $\mathbb{E}$ is non-zero:

$$1 = \Pr\left(d'|d\right) \frac{\beta u'(c[s',d'])}{u'(c[s,d])} \frac{1}{q(d'|d)} \tag{1}$$

where the rate of return on the state contingent claim is $1/q$.

The long derivation

FOC:

$$u'(c)\, q\left(d'|d\right) = \beta \Pr\left(d'|d\right) V_{y(d')}\left(s', d'\right)$$

Envelope:

$$\begin{aligned} V_{y(d)}(s,d) &= u'(c) && (2) \\ V_{y(d)}\left(s,\hat{d}\right) &= 0, \quad \hat{d} \neq d && (3) \end{aligned}$$

Note: only the y matching the realized value d has value.

$$u'(c[s,d])\, q\left(d'|d\right) = \beta \Pr\left(d'|d\right) u'\left(c[s',d']\right) \qquad (4)$$

Replicating Assets

Note how we can replicate any asset with a combination of state contingent claims:

- risk free bond = hold one claim for every d'
- capital = hold $R'(d')$ claims for every d'

Adding Bonds

Adding Bonds

We add bonds of different maturities to the Lucas model

In each period, maturity n bonds are issued.

- They pay 1 unit of consumption n periods from now.

Bonds issued earlier are traded in secondary markets.

- for maturities $i = 1, ..., n$.
- prices are $p_{t,i}$.

A maturity 1 bond becomes a maturity 0 bond next period.

- It pays 1 unit of consumption and then disappears.

Household Problem

State variables in period t:

- s_t shares
- $b_{t,i}$ bonds for $i = 0, ..., n-1$
- because a maturity $i+1$ bond purchased yesterday is a maturity i bond today.

Controls in period t:

- s_{t+1}: share purchases
- $b_{t+1,i}$ for $i = 0, ..., n-1$: bond purchases
- c_t: consumption

Dynamic Program

$$V(s, b_0, ..., b_{n-1}; d) = \max u(c) + \mathbb{E}\beta V(s', b'_0, ..., b'_{n-1}; d') \qquad (5)$$

subject to the budget constraint

$$c + \sum_{i=1}^{n} p_i b'_{i-1} + ps' = (p+d)s + \sum_{i=0}^{n-1} p_i b_i \qquad (6)$$

First-order conditions for stocks are standard.

First order conditions: bonds

$$b_i' : u'(c)p_{i+1} = \beta \mathbb{E} V_{b_i}(.') \tag{7}$$

Envelope:

$$V_{b_i} = u'(c)p_i \tag{8}$$

Euler equation:

$$u'(c)p_{i+1} = \beta \mathbb{E} u'(c')p_i' \tag{9}$$

Lucas asset pricing equation

$$1 = \mathbb{E}\left\{\frac{\beta u'(c')}{u'(c)}\frac{p_i'}{p_{i+1}}\right\} \tag{10}$$

The bond return is p_i'/p_{i+1}

- because buying a bond of maturity $i+1$ today gives a bond of maturity i tomorrow

Bond prices

The price sequence of a given bond is:

$$p_{n-1,t}, p_{n-2,t+1}, p_{n-3,t+2}, \ldots, 1 \tag{11}$$

Solve this by backward induction:

$$p_0 = 1 \tag{12}$$

Sub that into the Euler equation and iterate to find

$$p_{t,i} = \beta^i \mathbb{E} \frac{u'(c_{t+i})}{u'(c_t)} \tag{13}$$

with $c_t = d_t$.

Bond prices

These are actually the standard Lucas asset pricing equations

The per period return on the bond is

$$1 + r_{t,i} = (1/p_{t,i})^{1/i} \tag{14}$$

because a bond with constant interest rate $r_{t,i}$ for i periods would have the same payout.

Therefore:

$$u'(c_t) = \beta^i \mathbb{E} u'(c_{t+i})(1 + r_{t,i})^i \tag{15}$$

$r_{t,i}$ is not stochastic and $Eu'(c_{t+i}) = Eu'(d_{t+i})$ does not depend on the current state d.

Yield curve

Yield: the bond's average interest rate over its remaining life

$$1 + r_{t,i} = \left[u'(c_t)/Eu'(c_{t+i})\right]^{1/i}/\beta \tag{16}$$

Simple case: i.i.d. consumption.

In a "good period"

$$u'(c_t)/\mathbb{E}u'(c_{t+i}) = \gamma < 1 \tag{17}$$

One period bond:

$$1 + r_{t,1} = \gamma/\beta < 1/\beta \tag{18}$$

n period bond:

$$1 + r_{t,n} = \gamma^{1/n}/\beta \to 1/\beta \tag{19}$$

as $n \to \infty$.

The yield curve is upward sloping.

Yield curve intuition

An upward sloping yield curve predicts rising growth in the future.

Short yields are low when short term growth rate is low:

- households want to save to offset future low consumption
- low interest rates are needed to prevent saving
- in the example: growth from t to $t+1$ is negative

Long yields are high when long term growth rate is high:

- in the example: growth from $t+1$ to $t+2$ is 0

This is consistent with data (inverted yield curve "predicts" slow growth).

Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.

References I

Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.