

# Asset Pricing: Extensions

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Econ720

December 5, 2018

## State contingent claims

- ▶ Some assets pay out only in particular states of the world
  - ▶ e.g. insurance contracts
- ▶ Standard asset pricing formulas apply to those assets.
- ▶ It just adds notation...

# State contingent claims

- ▶ We start from the Lucas fruit tree model.
- ▶ In addition to stocks and bonds, households can purchase assets that pay out in exactly one state of the world.
  - ▶ Arrow securities or state contingent claims
- ▶ Their role in theory:
  - ▶ given a sufficiently rich set of Arrow securities, we can replicate any asset
  - ▶ can set up a model with all possible insurance opportunities (complete markets)

# Notation

- ▶ quantity purchased of asset that pays out in state  $d'$ :  $y'(d'|d)$ .
  - ▶ for convenience just write  $y(d')$
- ▶ price of that asset:  $q(d'|d)$ .

# Household

States: all assets held,  $k, b$ , and all  $y(d)$ .

▶ call that  $s$

Choices:  $b', k', y(d')$  for all  $d'$ .

Dynamic Program:

$$V(s, d) = \max_{c, k', b', y(d')} u(c) + \beta EV(s', d')$$

subject to

$$Rb + (p + d)k + y(d) = c + b' + pk' + \sum_{d'} q(d'|d) y'(d')$$

Note: only the  $y$  matching the realized value of  $d$  pays out.

## First-order conditions for state contingent claims

$$u'(c)q(d'|d) = \beta \Pr(d'|d) V_{y(d')} (s', d')$$

Envelope:

$$V_{y(d)}(s, d) = u'(c) \quad (1)$$

$$V_{y(d)}(s, \hat{d}) = 0, \hat{d} \neq d \quad (2)$$

Note: only the  $y$  matching the realized value  $d$  has value.

## Euler equation

$$u'(c[s, d])q(d'|d) = \beta \Pr(d'|d) u'(c[s', d']) \quad (3)$$

In more standard form:

$$1 = \Pr(d'|d) \frac{\beta u'(c[s', d'])}{u'(c[s, d])} \frac{1}{q(d'|d)} \quad (4)$$

where the rate of return on the state contingent claim is  $1/q$ .

## Lucas Equation

We could have written this down without any derivation by just applying the Lucas asset pricing equation:

$$1 = \mathbb{E}\left\{MRS_{t+1} \frac{\mathbb{I}(d')}{q(d', d)}\right\}$$

Special feature of Arrow securities: Only one term in the  $\mathbb{E}$  is non-zero.



Adding Bonds

## Adding Bonds

- ▶ We add bonds of different maturities to the Lucas model
- ▶ There are bonds for maturities  $i = 1, \dots, n$ .
- ▶ A bond of maturity  $i$  pays one unit of consumption  $i$  periods from now. Its price is  $p_{t,i}$ .
- ▶ These are discount bonds which do not pay interest.

# Household Problem

Controls in period  $t$ :

- ▶  $s_{t+1}$ : share purchases
- ▶  $b_{t+1,i}$  for  $i = 0, \dots, n-1$ : bond purchases

$c_t$ : consumption

State variables:  $s_t, b_{t,i}$  for  $i = 0, \dots, n-1$

Budget constraint:

$$c + \sum_{i=1}^n p_i b'_{i-1} + p s' = (p + d) s + \sum_{i=0}^{n-1} p_i b_i \quad (5)$$

Buying a maturity  $i$  bond today (at price  $p_i$ ) yields a maturity  $i-1$  bond tomorrow.

## Dynamic Program

$$V(s, b_0, \dots, b_{n-1}; d) = \max u(c) + \mathbb{E}\beta V(s', b'_0, \dots, b'_{n-1}; d')$$

subject to the budget constraint

First-order conditions:

Standard for the stocks, which yields the usual asset pricing equation.

For the bond:

$$b'_i : u'(c)p_{i+1} = \beta \mathbb{E}V_{b_i}(\cdot) \quad (6)$$

Envelope:

$$V_{b_i} = u'(c)p_i \quad (7)$$

## Euler equation

$$u'(c)p_{i+1} = \beta \mathbb{E} u'(c') p'_i$$

or, as Lucas asset pricing equation:

$$1 = \mathbb{E} \left\{ \frac{\beta u'(c')}{u'(c)} \frac{p'_i}{p_{i+1}} \right\} \quad (8)$$

The bond return is  $p'_i/p_{i+1}$

- ▶ because buying a bond of maturity  $i+1$  today gives a bond of maturity  $i$  tomorrow

The price sequence of a given bond is:

- ▶  $p_{n-1,t}, p_{n-2,t+1}, p_{n-3,t+2}, \dots, 1$

## Bond prices

Solve this by backward induction:

$$p_0 = 1 \quad (9)$$

Sub that into the Euler equation and iterate to find

$$p_{t,i} = \beta^i \mathbb{E} \frac{u'(c_{t+i})}{u'(c_t)} \quad (10)$$

with  $c_t = d_t$ .

## Bond prices

These are actually the standard Lucas asset pricing equations

The per period return on the bond is  $1 + r_{t,i} = (1/p_{t,i})^{1/i}$

Therefore:

$$u'(c_t) = \beta^i \mathbb{E} u'(c_{t+i}) (1 + r_{t,i})^i \quad (11)$$

$r_{t,i}$  is not stochastic and  $\mathbb{E} u'(c_{t+i}) = \mathbb{E} u'(d_{t+i})$  does not depend on the current state  $d$ .

## Yield curve

- ▶ Yield:  $1 + r_{t,i} = [u'(c_t) / \mathbb{E}u'(c_{t+i})]^{1/i} / \beta$
- ▶ With iid dividends: high consumption implies low yields for all maturities
- ▶ When  $c$  is above average ( $u'(c_t) < \mathbb{E}u'(c_{t+i})$ ), the yield curve is downward sloping
- ▶ This is consistent with data (the yield curve “predicts” slow growth).



# Reading

- ▶ Romer (2011), ch. 7.5
- ▶ Ljungqvist and Sargent (2004), ch. 7.

## References I

Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.