Asset Pricing: Extensions

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Econ720

November 15, 2019
State contingent claims

- Some assets pay out only in particular states of the world
  - e.g. insurance contracts
- Standard asset pricing formulas apply to those assets.
- It just adds notation...
State contingent claims

- We start from the Lucas fruit tree model.
- In addition to stocks and bonds, households can purchase assets that pay out in exactly one state of the world.
  - Arrow securities or state contingent claims
- Their role in theory:
  - given a sufficiently rich set of Arrow securities, we can replicate any asset
  - can set up a model with all possible insurance opportunities (complete markets)
Notation

- quantity purchased of asset that pays out in state $d'$: $y'(d'|d)$.
- for convenience just write $y(d')$
- price of that asset: $q(d'|d)$. 
Household

States: all assets held, $k, b$, and all $y(d)$.

▶ call that $s$

Choices: $b', k', y(d')$ for all $d'$.

Dynamic Program:

$$V(s, d) = \max_{c, k', b', y(d')} u(c) + \beta EV(s', d')$$

subject to

$$Rb + (p + d)k + y(d) = c + b' + pk' + \sum_{d'} q(d'|d) y'(d')$$

Note: only the $y$ matching the realized value of $d$ pays out.
First-order conditions for state contingent claims

\[ u'(c) q(d'|d) = \beta \Pr(d'|d) V_{y(d')} (s', d') \]

Envelope:

\[ V_{y(d)} (s, d) = u'(c) \quad (1) \]
\[ V_{y(d)} (s, \hat{d}) = 0, \quad \hat{d} \neq d \quad (2) \]

Note: only the \( y \) matching the realized value \( d \) has value.
Euler equation

\[ u'(c[s,d])q(d'|d) = \beta \Pr(d'|d) u'(c[s',d']) \]  \hspace{1cm} (3)

In more standard form:

\[ 1 = \Pr(d'|d) \frac{\beta u'(c[s',d'])}{u'(c[s,d])} \frac{1}{q(d'|d)} \]  \hspace{1cm} (4)

where the rate of return on the state contingent claim is \(1/q\).
We could have written this down without any derivation by just applying the Lucas asset pricing equation:

$$1 = \mathbb{E}\left\{ MRS_{t+1} \frac{\Pi(d')}{q(d', d)} \right\}$$

Special feature of Arrow securities: Only one term in the $\mathbb{E}$ is non-zero.
Adding Bonds
Adding Bonds

- We add bonds of different maturities to the Lucas model.
- There are bonds for maturities $i = 1, \ldots, n$.
- A bond of maturity $i$ pays one unit of consumption $i$ periods from now. Its price is $p_{t,i}$.
- These are discount bonds which do not pay interest.
Household Problem

Controls in period $t$:

- $s_{t+1}$: share purchases
- $b_{t+1,i}$ for $i = 0, \ldots, n - 1$: bond purchases

$c_t$: consumption

State variables: $s_t$, $b_{t,i}$ for $i = 0, \ldots, n - 1$

Budget constraint:

$$c + \sum_{i=1}^{n} p_i b'_{i-1} + ps' = (p + d)s + \sum_{i=0}^{n-1} p_i b_i$$

(5)

Buying a maturity $i$ bond today (at price $p_i$) yields a maturity $i - 1$ bond tomorrow.
Dynamic Program

\[ V(s, b_0, ..., b_{n-1}; d) = \max u(c) + \mathbb{E} \beta V(s', b'_0, ..., b'_{n-1}; d') \]

subject to the budget constraint

First-order conditions:
Standard for the stocks, which yields the usual asset pricing equation.
For the bond:

\[ b'_i : u'(c)p_{i+1} = \beta \mathbb{E} V_{b_i}(\cdot') \]  \hspace{1cm} (6)

Envelope:

\[ V_{b_i} = u'(c)p_i \]  \hspace{1cm} (7)
Euler equation

\[ u'(c)p_{i+1} = \beta \mathbb{E} u'(c')p'_i \]

or, as Lucas asset pricing equation:

\[ 1 = \mathbb{E} \left\{ \frac{\beta u'(c')}{u'(c)} \frac{p'_i}{p_{i+1}} \right\} \quad (8) \]

The bond return is \( p'_i/p_{i+1} \)

because buying a bond of maturity \( i + 1 \) today gives a bond of maturity \( i \) tomorrow

The price sequence of a given bond is:

\[ p_{n-1}, t, p_{n-2}, t+1, p_{n-3}, t+2, \ldots, 1 \]
Solve this by backward induction:

\[ p_0 = 1 \]  \hspace{1cm} (9)

Sub that into the Euler equation and iterate to find

\[ p_{t,i} = \beta^i \mathbb{E} \frac{u'(c_{t+i})}{u'(c_t)} \]  \hspace{1cm} (10)

with \( c_t = d_t \).
These are actually the standard Lucas asset pricing equations.
The per period return on the bond is \( 1 + r_{t,i} = (1/p_{t,i})^{1/i} \)

Therefore:

\[
u'(c_t) = \beta^i \mathbb{E}u'(c_{t+i})(1 + r_{t,i})^i \quad (11)\]

\( r_{t,i} \) is not stochastic and \( \mathbb{E}u'(c_{t+i}) = \mathbb{E}u'(d_{t+i}) \) does not depend on the current state \( d \).
Yield curve

- Yield: \( 1 + r_{t,i} = \left[ u'(c_t) / E u'(c_{t+i}) \right]^{1/i} / \beta \)
- With iid dividends: high consumption implies low yields for all maturities
- When \( c \) is above average \( (u'(c_t) < E u'(c_{t+i})) \), the yield curve is downward sloping
- This is consistent with data (the yield curve “predicts” slow growth).
Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.
References I
