

Stochastic Growth Model

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Introduction

We now return to the stochastic growth model.

We study

- ▶ the planner's problem
- ▶ the competitive equilibrium

Then we introduce heterogeneity and risk sharing.

Planning solution

The history of shocks is θ^t .

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \quad (1)$$

Technology:

$$X = F(K, L, \theta) + (1 - \delta)K - c \quad (2)$$

$$K' = X \quad (3)$$

Bellman equation

Define $k = K/L$.

$$V(k, \theta) = \max_{k' \in [0, f(k, \theta) + (1 - \delta)k]} u(f(k, \theta) + (1 - \delta)k - k') \quad (4)$$

$$+ \beta E[V(k', \theta') | \theta] \quad (5)$$

First-order conditions

- ▶ Verify that A1-A5 hold ... Theorems 1-6 apply.

- ▶ FOC

$$u'(c) = \beta EV_k(k', \theta')$$

- ▶ Envelope

$$V_k(k, \theta) = u'(c) [f_k(k, \theta) + 1 - \delta]$$

- ▶ Euler

$$u'(c) = \beta E [u'(c') \{f_k(k', \theta') + 1 - \delta\} | \theta] \quad (6)$$

- ▶ Solution: $V(k, \theta)$ and $\pi(k, \theta)$ that "solve" the Bellman equation

Characterization

- ▶ Now for the bad news ... there really isn't much one can say about the solution analytically.
- ▶ But see Campbell (1994) for a discussion of a log-linear approximation.

Competitive Equilibrium

Competitive equilibrium

The model comes in 2 flavors.

1. Complete markets

- ▶ for every history, there exists an asset that pays in that state of the world
- ▶ the implication is complete risk sharing: all idiosyncratic risks are insured
- ▶ aggregate risks remain

2. Incomplete markets

- ▶ some securities are missing
- ▶ there is no representative agent

Trading arrangements

- ▶ With complete markets, date 1 Arrow-Debreu trading is convenient
 - ▶ Uncertainty essentially disappears from the model.
- ▶ With incomplete markets, it is easiest to specify the set of securities available at each date.
 - ▶ Sequential trading.

Complete markets - Arrow Debreu trading

- ▶ The environment is standard.
- ▶ The history is of shocks is θ^t .
- ▶ Trading takes place at date 1.
- ▶ The point: This looks like a static model without uncertainty.

Market arrangements

Goods markets: standard

- ▶ buy and sell consumption at each node θ^t
- ▶ price $p(\theta^t)$

Labor markets: standard

- ▶ wage $w(\theta^t)$

Capital rental:

- ▶ households can buy goods in θ^t and give them to firms
- ▶ firms then pay $R(\theta^{t+1})$ tomorrow
- ▶ this includes returning the undepreciated capital

Household: budget constraint

Expenditures in state θ^t :

$$x(\theta^t) = p(\theta^t)[c(\theta^t) + s(\theta^t)] \quad (7)$$

$p(\theta^t)$ is the price of the good in state θ^t .

c is consumption

s is "saving:" buy goods (capital) and rent to firms.

Household: budget constraint

Income in state θ^t :

$$y(\theta^t) = w(\theta^t) + R(\theta^t)s(\theta^{t-1}) \quad (8)$$

$w(\theta^t)$ is the wage.

$R(\theta^t)$ is the payoff from renting a unit of the good to the firm.

Both are state contingent.

Poor notation: keep in mind that θ^t follows θ^{t-1}

Household: budget constraint

Lifetime budget constraint:

$$\sum_{t=0}^{\infty} \sum_{\theta^t} [y(\theta^t) - x(\theta^t)] + p(\theta_0) s_0 = 0 \quad (9)$$

s_0 is the initial endowment of goods.

With Arrow-Debreu trading, there is a lifetime budget constraint, even under uncertainty.

- ▶ Because there really is no uncertainty any more.
- ▶ At each node, the household's spending and income are fully predictable.

Firms

Firms maximize the total value of profits.

- ▶ There is no discounting because of Arrow-Debreu trading.

Profits in state θ^t :

$$p(\theta^t)[F(K[\theta^t], L[\theta^t], \theta_t) + (1 - \delta)K[\theta^t]] \\ - R(\theta^t)K(\theta^t) - w(\theta^t)L(\theta^t)$$

Value of the firm: sum of profits over all states.

FOCs are standard:

- ▶ since the firm does not own anything, it maximizes profits state-by-state.

Competitive Equilibrium

- ▶ Allocation: $c(\theta^t), s(\theta^t), K(\theta^t), L(\theta^t)$.
- ▶ Price system: $p(\theta^t), w(\theta^t), R(\theta^t)$ for all histories θ^t .
- ▶ These satisfy:
 1. Household optimality.
 2. Firm optimality.
 3. Market clearing:
 - ▶ $L(\theta^t) = 1$.
 - ▶ $K(\theta^t, \theta_{t+1}) = s(\theta^t)$.
 - ▶ Goods market.

Competitive Equilibrium

Comments

- ▶ This looks like a static model without uncertainty.
 - ▶ Each history defines new goods: output, labor, capital rental.
- ▶ The setup is far more complicated than the recursive one.

Risk Sharing

- ▶ What if agents are heterogeneous?
- ▶ With complete markets, risk is perfectly shared.
- ▶ The simplest case: An endowment economy with Arrow-Debreu trading.
- ▶ The state is θ^t .

Risk Sharing

Households

- ▶ There are I types of households, indexed by i .
- ▶ Endowments are $y^i(\theta^t)$.
- ▶ Preferences are

$$\sum_t \sum_{\theta^t} \beta^t q(\theta^t) u^i(c^i[\theta^t])$$

- ▶ Budget constraints:

$$\sum_t \sum_{\theta^t} p(\theta^t) [c^i(\theta^t) - y^i(\theta^t)] = 0 \quad (10)$$

Risk Sharing

Households

First-order conditions are as usual:

$$q(\theta^t) \beta^t \frac{\partial u^i(c^i[\theta^t])}{\partial c^i[\theta^t]} = \lambda_i p(\theta^t) \quad (11)$$

where λ_i is the Lagrange multiplier.

Risk Sharing

Complete risk sharing: For all θ^t the MRS is equated across households:

$$MRS(\theta^t, \hat{\theta}^\tau) = -\frac{\beta^t \partial u^i(c^i[\theta^t]) / \partial c^i[\theta^t]}{\beta^\tau \partial u^i(c^i[\hat{\theta}^\tau]) / \partial c^i[\hat{\theta}^\tau]} = \frac{p(\theta^t) / q(\theta^t)}{p(\hat{\theta}^\tau) / q(\hat{\theta}^\tau)}$$

Equivalently, the ratio of marginal utilities between 2 agents is the same for all θ^t :

$$\frac{\partial u^i(c^i[\theta^t]) / \partial c^i[\theta^t]}{\partial u^j(c^j[\theta^t]) / \partial c^j[\theta^t]} = \frac{\lambda_i}{\lambda_j} \quad (12)$$

Implications

Individual consumption still fluctuates because the aggregate endowment changes over time.

- ▶ aggregate risk cannot be insured

If there is no aggregate uncertainty, then individual consumption is constant.

Proof:

$$\partial u^i / \partial c^i = (\lambda_i / \lambda_1) \partial u^1 / \partial c^1 \quad (13)$$

That implies an increasing function $c^i = f_i(c^1)$ that is the same for all states θ^t .

Market clearing: $\sum_i c^i = \sum_i f_i(c^1) = y$.

This has a unique solution c^1 . \square

Sequential Trading

Sequential Trading

- ▶ We set up the C.E. with sequential trading.
- ▶ If we want complete markets, we need **Arrow securities**.
- ▶ Each security, $a(\theta^{t+1})$ is indexed by the state of the world in which it pays off: θ^{t+1} .
- ▶ The asset is purchased for price $\bar{p}(\theta^t, \theta')$ in state θ^t .
- ▶ It pays one unit of consumption if $\theta^{t+1} = [\theta^t, \theta']$.

Household

- ▶ Budget constraint:

$$c(\theta^t) + s(\theta^t) = w(\theta^t) + a(\theta^t) + R(\theta^t)k(\theta^t) \quad (14)$$

$$s(\theta^t) = \sum_{\theta_{t+1}} \bar{p}(\theta^t, \theta_{t+1}) a(\theta^t, \theta_{t+1}) + x(\theta^t) \quad (15)$$

$$k(\theta^t, \theta_{t+1}) = x(\theta^t) \quad (16)$$

- ▶ Numeraire: consumption at each node θ^t .

Household

- ▶ Household problem:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \quad (17)$$

s.t. budget constraints for all θ^t .

Recursive household problem

- ▶ State: (\vec{a}, k, θ) .
 - ▶ \vec{a} : holdings of all the $a(\theta)$.
- ▶ Given prices: w and $\bar{p}(\theta, \theta')$.
- ▶ Bellman equation:

$$V(\vec{a}, k, \theta) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V(\vec{a}', k', \theta')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w + a(\theta) + Rk$$

First order conditions

- ▶ For $a'(\theta')$:

$$u'(c)\bar{p}(\theta, \theta') = \beta q(\theta'|\theta) \frac{\partial V(\vec{a}'[\theta'], k', \theta')}{\partial a(\theta')} \quad (18)$$

- ▶ For k' :

$$u'(c) = \beta \sum_{\theta'} q(\theta'|\theta) \frac{\partial V(\vec{a}', k', \theta')}{\partial k'} \quad (19)$$

First order conditions

- ▶ Envelope:

$$\partial V(\vec{a}, k, \theta) / \partial a(\theta) = u'(c) \quad (20)$$

$$\partial V(\vec{a}, k, \hat{\theta}) / \partial a(\theta) = 0 \quad (21)$$

$$\partial V(\vec{a}, k, \theta) / \partial k = u'(c)R \quad (22)$$

- ▶ Euler equation holds state by state for state contingent claims:

$$u'(c)\bar{p}(\theta, \theta') = \beta q(\theta'|\theta) u'(c[a'(\theta'), \theta']) \quad (23)$$

- ▶ Euler equation for capital:

$$\begin{aligned} u'(c) &= \beta \sum_{\theta'} q(\theta'|\theta) R(\theta, \theta') u'(c[a'(\theta'), k', \theta']) \quad (24) \\ &= \beta E R' u'(c') \end{aligned}$$

No arbitrage

- ▶ Since capital can be replicated by buying a set of Arrow securities:

$$\sum_{\theta'} \bar{p}(\theta, \theta') R(\theta, \theta') = 1 \quad (25)$$

- ▶ Proof: Solve (23) for $q(\theta'|\theta)$ and substitute into (24).

Equilibrium

- ▶ We can write down a sequential equilibrium definition, similar to the Arrow-Debreu.
 - ▶ Everything is indexed by θ^t .
- ▶ More powerful: Recursive Competitive Equilibrium.
 - ▶ Everything is a function of the current state.

Recursive CE

- ▶ Define an aggregate state vector: $S = (\theta, K)$.
 - ▶ In general: we need to keep track of the distribution of (θ_i, k_i) across households.
 - ▶ Here: all households are identical.
- ▶ The law of motion for the aggregate state:

$$\begin{aligned}\Pr(\theta'|\theta) &= q(\theta'|\theta) \\ K' &= G(\theta, K)\end{aligned}$$

where G is endogenous.

Recursive CE

Household

- ▶ Given:
 - ▶ aggregate state and its law of motion.
 - ▶ price functions: $w(S)$, $R(S)$ and $\bar{p}(S, \theta')$.
- ▶ Bellman equation:

$$V(\vec{a}, k, S) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta' | \theta) V(\vec{a}'[\theta'], k', S')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w(S) + a(\theta) + R(S)k$$

and aggregate law of motion

$$S' = G(S)$$

Recursive CE

- ▶ First-order conditions: unchanged.
- ▶ Solution: $V(a, k, S)$ and policy functions $c(a, k, S)$,
 $k' = \kappa(a, k, S)$.

Recursive CE

Firm

- ▶ Always the same because the firm has a static problem:
- ▶ Solution: $R(S), w(S)$.

Recursive CE

- ▶ Equilibrium objects:
 1. Household: Value function and policy functions.
 2. Firm: Price functions.
 3. Aggregate law of motion: $K' = G(\theta, K)$.
- ▶ Equilibrium conditions:
 1. Household optimality.
 2. Firm optimality.
 3. Market clearing.
 4. Consistency:

$$G(\theta, K) = \kappa(K, \theta, K) \quad (26)$$

where the household's policy function is $k' = \kappa(k, \theta, K)$.

Recursive CE

- ▶ Note: We could toss out all the Arrow securities without changing anything.
- ▶ The model boils down to:
 1. Euler equation for K : $u'(c) = \beta E[R' u'(c')]$
 2. Law of motion for K : $K' = F(K, L) + (1 - \delta)K - c$.
 3. FOC: $R = F_K(K, L) + 1 - \delta$.
- ▶ This changes when individuals are not identical.

Recursive CE

What do we gain?

- ▶ Avoid having to carry around infinite histories.
- ▶ Equilibrium contains few objects.
 - ▶ Especially when the economy is **stationary**.
- ▶ All endogenous objects are functions.
 - ▶ Results from functional analysis can be used to determine their properties.
- ▶ Recursive CE is easy to compute.

Reading

- ▶ Acemoglu (2009) ch. 16-17.
- ▶ Krusell (2014) ch. 6
- ▶ Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- ▶ Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains. Ch. 7 covers complete market economies (Arrow-Debreu and sequential trading). Ch. 6: Recursive CE.
- ▶ Campbell (1994) discusses an analytical solution (approximate)

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