Stochastic Growth Model

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We now return to the stochastic growth model.

We study

- the planner’s problem
- the competitive equilibrium

Then we introduce heterogeneity and risk sharing.
Planning solution

The history of shocks is $\theta^t$.

Preferences:

$$\sum_{i=0}^{\infty} \beta^i \sum_{\theta^t} \Pr(\theta^t|\theta_0) u(c[\theta^t])$$  

(1)

Technology:

$$X = F(K,L,\theta) + (1 - \delta)K - c$$  

(2)

$$K' = X$$  

(3)
Define $k = K/L$.

\[
V(k, \theta) = \max_{k' \in [0, f(k, \theta) + (1 - \delta) k]} u(f(k, \theta) + (1 - \delta) k - k') + \beta E[V(k', \theta') | \theta]
\] (4)

(5)
First-order conditions

- Verify that A1-A5 hold ... Theorems 1-6 apply.
- FOC

\[ u'(c) = \beta EV_k (k', \theta') \]

- Envelope

\[ V_k (k, \theta) = u'(c) [f_k (k, \theta) + 1 - \delta] \]

- Euler

\[ u'(c) = \beta E [u'(c') \{f_k (k', \theta') + 1 - \delta\} | \theta] \] (6)

- Solution: \( V(k, \theta) \) and \( \pi(k, \theta) \) that "solve" the Bellman equation
Now for the bad news ... there really isn’t much one can say about the solution analytically.

But see Campbell (1994) for a discussion of a log-linear approximation.
Competitive Equilibrium
The model comes in 2 flavors.

1. Complete markets
   - for every history, there exists an asset that pays in that state of the world
   - the implication is complete risk sharing: all idiosyncratic risks are insured
   - aggregate risks remain

2. Incomplete markets
   - some securities are missing
   - there is no representative agent
Trading arrangements

- With complete markets, date 1 Arrow-Debreu trading is convenient
  - Uncertainty essentially disappears from the model.
- With incomplete markets, it is easiest to specify the set of securities available at each date.
  - Sequential trading.
Complete markets - Arrow Debreu trading

- The environment is standard.
- The history is of shocks is $\theta^t$.
- Trading takes place at date 1.
- The point: This looks like a static model without uncertainty.
Market arrangements

Goods markets: standard

- buy and sell consumption at each node $\theta^t$
- price $p(\theta^t)$

Labor markets: standard

- wage $w(\theta^t)$

Capital rental:

- households can buy goods in $\theta^t$ and give them to firms
- firms then pay $R(\theta^{t+1})$ tomorrow
- this includes returning the undepreciated capital
Expenditures in state $\theta^t$:

$$x(\theta^t) = p(\theta^t)[c(\theta^t) + s(\theta^t)]$$

(7)

$p(\theta^t)$ is the price of the good in state $\theta^t$.

$c$ is consumption

$s$ is "saving:" buy goods (capital) and rent to firms.
Household: budget constraint

Income in state $\theta^t$:

$$y(\theta^t) = w(\theta^t) + R(\theta^t)s(\theta^{t-1})$$  \hspace{1cm} (8)

$w(\theta^t)$ is the wage.

$R(\theta^t)$ is the payoff from renting a unit of the good to the firm.

Both are state contingent.

Poor notation: keep in mind that $\theta^t$ follows $\theta^{t-1}$
Lifetime budget constraint:

\[
\sum_{t=0}^{\infty} \sum_{\theta^t} [y(\theta^t) - x(\theta^t)] + p(\theta_0)s_0 = 0
\]  

(9)

\(s_0\) is the initial endowment of goods.

With Arrow-Debreu trading, there is a lifetime budget constraint, even under uncertainty.

- Because there really is no uncertainty any more.
- At each node, the household’s spending and income are fully predictable.
Firms

Firms maximize the total value of profits.

- There is no discounting because of Arrow-Debreu trading.

Profits in state $\theta^t$:

$$p(\theta^t)[F(K[\theta^t], L[\theta^t], \theta_t) + (1 - \delta)K[\theta^t]] - R(\theta^t)K(\theta^t) - w(\theta^t)L(\theta^t)$$

Value of the firm: sum of profits over all states.

FOCs are standard:

- since the firm does not own anything, it maximizes profits state-by-state.
Competitive Equilibrium

- Allocation: $c(\theta^t), s(\theta^t), K(\theta^t), L(\theta^t)$.
- Price system: $p(\theta^t), w(\theta^t), R(\theta^t)$ for all histories $\theta^t$.
- These satisfy:
  1. Household optimality.
  2. Firm optimality.
  3. Market clearing:
     - $L(\theta^t) = 1$.
     - $K(\theta^t, \theta_{t+1}) = s(\theta^t)$.
     - Goods market.
This looks like a static model without uncertainty.
  Each history defines new goods: output, labor, capital rental.
The setup is far more complicated than the recursive one.
Risk Sharing

- What if agents are heterogeneous?
- With complete markets, risk is perfectly shared.
- The simplest case: An endowment economy with Arrow-Debreu trading.
- The state is $\theta^t$. 
There are $I$ types of households, indexed by $i$.

Endowments are $y^i(\theta^t)$.

Preferences are

$$\sum_t \sum_{\theta^t} \beta^t q(\theta^t) u^i(c^i[\theta^t])$$

Budget constraints:

$$\sum_t \sum_{\theta^t} p(\theta^t) \left[ c^i(\theta^t) - y^i(\theta^t) \right] = 0 \quad (10)$$
First-order conditions are as usual:

\[ q(\theta^t) \beta^t \frac{\partial u^i(c^i[\theta^t])}{\partial c^i[\theta^t]} = \lambda_i p(\theta^t) \]  \hspace{1cm} (11)

where \( \lambda_i \) is the Lagrange multiplier.
Complete risk sharing: For all $\theta^t$ the MRS is equated across households:

$$MRS \left( \theta^t, \hat{\theta}^\tau \right) = -\frac{\beta^t \frac{\partial u^i \left( c^i [\theta^t] \right)}{\partial c^i [\theta^t]} \beta^\tau \frac{\partial u^i \left( c^i \left[ \hat{\theta}^\tau \right] \right)}{\partial c^i \left[ \hat{\theta}^\tau \right]}}{p(\theta^t)/q(\theta^t)} = \frac{p \left( \theta^t \right) / q \left( \theta^t \right)}{p \left( \hat{\theta}^\tau \right) / q \left( \hat{\theta}^\tau \right)}$$

Equivalently, the ratio of marginal utilities between 2 agents is the same for all $\theta^t$:

$$\frac{\partial u^i \left( c^i [\theta^t] \right)}{\partial c^i [\theta^t]} / \frac{\partial u^j \left( c^j [\theta^t] \right)}{\partial c^j [\theta^t]} = \frac{\lambda_i}{\lambda_j}$$ (12)
Implications

Individual consumption still fluctuates because the aggregate endowment changes over time.

- aggregate risk cannot be insured

If there is no aggregate uncertainty, then individual consumption is constant.

Proof:

\[
\frac{\partial u^i}{\partial c^i} = \left(\frac{\lambda_i}{\lambda_1}\right) \frac{\partial u^1}{\partial c^1}
\]  

(13)

That implies an increasing function \( c^i = f_i(c^1) \) that is the same for all states \( \theta^t \).

Market clearing: \( \sum_i c^i = \sum_i f_i(c^1) = y \).

This has a unique solution \( c^1 \). \( \square \)
Sequential Trading
Sequential Trading

- We set up the C.E. with sequential trading.
- If we want complete markets, we need Arrow securities.
- Each security, $a(\theta^{t+1})$ is indexed by the state of the world in which it pays off: $\theta^{t+1}$.
- The asset is purchased for price $\bar{p}(\theta^t, \theta')$ in state $\theta^t$.
- It pays one unit of consumption if $\theta^{t+1} = [\theta^t, \theta']$. 
Household

- Budget constraint:

\[
c(\theta^t) + s(\theta^t) = w(\theta^t) + a(\theta^t) + R(\theta^t)k(\theta^t) \quad (14)
\]

\[
s(\theta^t) = \sum_{\theta_{t+1}} \bar{p}(\theta^t, \theta_{t+1})a(\theta^t, \theta_{t+1}) + x(\theta^t) \quad (15)
\]

\[
k(\theta^t, \theta_{t+1}) = x(\theta^t) \quad (16)
\]

- Numeraire: consumption at each node $\theta^t$. 

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Household problem:

\[
\max \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t|\theta_0) u(c[\theta^t])
\]

s.t. budget constraints for all \( \theta^t \).
Recursive household problem

▶ State: \((\vec{a}, k, \theta)\).
  ▶ \(\vec{a}\): holdings of all the \(a(\theta)\).
▶ Given prices: \(w\) and \(\vec{p}(\theta, \theta')\).
▶ Bellman equation:

\[
V(\vec{a}, k, \theta) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V(\vec{a}', k', \theta')
\]

s.t. budget constraint

\[
\sum_{\theta'} \vec{p}(\theta, \theta') a'(\theta') + k' + c = w + a(\theta) + Rk
\]
First order conditions

For $a'(\theta')$:

$$
u'(c)\bar{p}(\theta, \theta') = \beta q(\theta'|\theta) \frac{\partial V(\overrightarrow{d'}[\theta'], k', \theta')}{\partial a(\theta')} \quad (18)$$

For $k'$:

$$
u'(c) = \beta \sum_{\theta'} q(\theta'|\theta) \frac{\partial V(\overrightarrow{d'}, k', \theta')}{\partial k'} \quad (19)$$
First order conditions

▶ Envelope:

\[
\begin{align*}
\frac{\partial V(\overrightarrow{a},k,\theta)}{\partial a(\theta)} &= u'(c) \quad (20) \\
\frac{\partial V(\overrightarrow{a},k,\hat{\theta})}{\partial a(\theta)} &= 0 \quad (21) \\
\frac{\partial V(\overrightarrow{a},k,\theta)}{\partial k} &= u'(c)R \quad (22)
\end{align*}
\]

▶ Euler equation holds state by state for state contingent claims:

\[
u'(c)\bar{p}(\theta,\theta') = \beta q(\theta'|\theta) \ u'(c [a'(\theta'), \theta']) \quad (23)
\]

▶ Euler equation for capital:

\[
\begin{align*}
u'(c) &= \beta \sum_{\theta'} q(\theta'|\theta) R(\theta,\theta') u'(c [a'(\theta'), k', \theta']) \quad (24) \\
&= \beta E R' u'(c')
\end{align*}
\]
No arbitrage

Since capital can be replicated by buying a set of Arrow securities:

\[
\sum_{\theta'} \bar{p} (\theta, \theta') R (\theta, \theta') = 1
\]  

(25)

Proof: Solve (23) for \( q(\theta' | \theta) \) and substitute into (24).
Equilibrium

- We can write down a sequential equilibrium definition, similar to the Arrow-Debreu.
  - Everything is indexed by $\theta^t$.
  - Everything is a function of the current state.
Define an aggregate state vector: $S = (\theta, K)$. In general: we need to keep track of the distribution of $(\theta_i, k_i)$ across households. Here: all households are identical.

The law of motion for the aggregate state:

$$
\Pr(\theta'|\theta) = q(\theta'|\theta)
$$

$$
K' = G(\theta, K)
$$

where $G$ is endogenous.
Recursive CE

Household

- Given:
  - aggregate state and its law of motion.
  - price functions: \( w(S), R(S) \) and \( \bar{p}(S, \theta') \).

- Bellman equation:

\[
V(\overline{\theta}, k, S) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta' | \theta) V(\overline{\theta'} [\theta'], k', S')
\]

s.t. budget constraint

\[
\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w(S) + a(\theta) + R(S) k
\]

and aggregate law of motion

\[
S' = G(S)
\]
First-order conditions: unchanged.

Solution: \( V(a, k, S) \) and policy functions \( c(a, k, S) \), \( k' = \kappa(a, k, S) \).
Recursive CE
Firm

- Always the same because the firm has a static problem:
- Solution: $R(S), w(S)$.  

Recursive CE

- **Equilibrium objects:**
  1. Household: Value function and policy functions.
  3. Aggregate law of motion: $K' = G(\theta, K)$.

- **Equilibrium conditions:**
  1. Household optimality.
  2. Firm optimality.
  4. Consistency:

\[
G(\theta, K) = \kappa(K, \theta, K) \tag{26}
\]

where the household’s policy function is $k' = \kappa(k, \theta, K)$. 

Note: We could toss out all the Arrow securities without changing anything.

The model boils down to:

1. Euler equation for $K$: \[ u'(c) = \beta E[R'u'(c')] \]
2. Law of motion for $K$: \[ K' = F(K, L) + (1 - \delta) K - c. \]
3. FOC: \[ R = F_K(K, L) + 1 - \delta. \]

This changes when individuals are not identical.
Recursive CE
What do we gain?

▶ Avoid having to carry around infinite histories.
▶ Equilibrium contains few objects.
  ▶ Especially when the economy is **stationary**.
▶ All endogenous objects are functions.
  ▶ Results from functional analysis can be used to determine their properties.
▶ Recursive CE is easy to compute.
Reading

- Acemoglu (2009) ch. 16-17.
- Krusell (2014) ch. 6
- Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains. Ch. 7 covers complete market economies (Arrow-Debreu and sequential trading). Ch. 6: Recursive CE.
- Campbell (1994) discusses an analytical solution (approximate)
References I


