Review Questions: Asset Pricing
Econ720. Fall 2019. Prof. Lutz Hendricks

1 Many Trees

Consider the standard Lucas fruit tree model. Assume that household \( i \) initially holds \( K_i \) trees of type \( i \). Each tree produces a dividend \( d_{it} \) that is independently drawn from some distribution \( G(d) \).

1. Define a competitive equilibrium.
2. Characterize the optimal portfolio held by each household.
3. Show that every household enjoys constant consumption, even without having access to Arrow securities.

Note: The answer is kind of obvious. I have not formally derived it. So I don’t know how hard that is.

2 Lucas Trees

Consider a version of Lucas’s asset pricing model with one representative household who is endowed with one tree.

**Technology:** The tree produces a stream of dividends, \( d_t \), where \( d_0 = 1 \). The dividend growth rate, \( \frac{d_{t+1}}{d_t} \), can take on one of two values, \( \mu + \sigma \) or \( \mu - \sigma \), where \( \mu > 1 \). The dividend growth rate is a Markov chain with transition matrix \( P \). In particular, assume \( P \) is a symmetric matrix where the probability of switching growth rates is \( p \), where \( p \in (0, 1) \).

**Preferences:** Household preferences are given by \( E \sum_{t=0}^{\infty} \beta^t \ln (c_t) \).

**Markets:** At each date, there are markets for consumption goods, trees, and state-contingent claims that pay one unit of consumption tomorrow in a particular state of the world.

**Questions:**
(a) Define a solution to the household problem. Think carefully about what the household’s state variables are.
(b) Define a recursive competitive equilibrium.
(c) Solve for the equilibrium pricing function for trees. You should find that the price-dividend ratio for trees is constant over time.
(d) Solve for the pricing functions for state-contingent claims. (Assume \( \lim_{n \to \infty} \beta^{n+1} E_t x_{t+n} = 0 \) to rule out bubbles).
(e) Add a riskless bond to this economy (a sure claim to one unit of the consumption good next period). Compute the price of a riskless bond. Hint: There is no need to resolve for the equilibrium price functions (why not?).
(f) Now assume \( p = 0.5 \). Compute the average rate of return on bonds and trees. What is the equity premium for this economy?
2.1 Answer: Lucas Trees

(a) Let \( \mu(z) = \frac{d}{dz} \). The household’s state variables are the exogenous states \((z, d)\) and last period’s choices \(s, y(0), y(1)\). The Bellman equation is

\[
v(s, y(0), y(1), z, d) = \max_{s', y'(0), y'(1)} \{ \ln c + \beta E[v(s', y'(0), y'(1), z', d')] \}
\]

subject to the budget constraint

\[
c + p(z, d) s' + y'(1) q(1|z) + y'(0) q(0|z) = s(p(z, d) + d) + y(z)
\]

\[
d' = \mu(z') d
\]

More explicitly

\[
v(s, y(0), y(1), z, d) = \max_u \left( s(p(z, d) + d) + y(z) - p(z, d) s' - \sum_{z'} y'(z') q(z'|z) \right) + \beta \sum_{z'} \Pr(z'|z) v(s', y'(0), y'(1), z', d\mu(z'))
\]

FOCs:

\[
p(z, d) u'(c) = \beta \sum_{z'} \Pr(z'|z) v_1(s', y'(0), y'(1), z', d\mu(z'))
\]

\[
q(z'|z) u'(c) = \beta \Pr(z'|z) v_{y'(z')}(s', y'(0), y'(1), z', d\mu(z'))
\]

where the latter holds for all \(z'\). Envelope conditions:

\[
v_1(.) = u'(c) (p(z, d) + d)
\]

\[
v_{y(z')}(.) = u'(c)
\]

The Euler equation is standard:

\[
u'(c) p(z, d) = \beta \sum_{z'} \Pr(z'|z) u'(c') [p(z', d\mu(z')) + d\mu(z')]
\]

There is another Euler equation for state-contingent claims:

\[
u'(c) q(z'|z) = \beta \Pr(z'|z) u'(c')
\]

where it is understood that \(c'\) is the realization in the right state tomorrow.

A solution to the household problem consists of a value function and policy functions for \(s' (s, y(0), y(1), z, d)\) and \(q'(z'; s, y(0), y(1), z, d)\) such that:

- \(v\) satisfies the fixed-point property of the Bellman equation, given optimal policies.
- Optimal policies maximize the right hand side of the Bellman equation, given \(v\).

(b) A recursive competitive equilibrium is:

1. A set of individual decision rules,
2. A set of pricing functions \(p(z, d), q(z'|z)\),

such that:

1. Given pricing functions the decision rules solve the household’s dynamic program (see (a)).

2. Markets clear:

\[
s'(1, 0, 0, z, d) = 1
\]

\[
y'(z'; z, 0, 0, z, d) = 0
\]

2
(c) It’s convenient to use time subscripts now:

\[
\frac{p_t}{c_t} = \beta E_t \left[ \frac{1}{c_{t+1}} (p_{t+1} + d_{t+1}) \right]
\]

In equilibrium \( c_t = d_t \). Hence,

\[
\frac{p_t}{d_t} = \beta E_t \left[ \frac{1}{d_{t+1}} (p_{t+1} + d_{t+1}) \right]
\]

Define \( x = p/d \). Iterating forward and using law of iterated expectations then yields

\[
x_t = \beta + \beta^2 + \beta^3 + \ldots + \beta^{n+1} E_{t} E_{t+1} \ldots E_{n-1} \left[ E_n \{ x_n \} \right] 
\]

Assuming \( \lim_{n \to \infty} \beta^{n+1} E_t x_{t+n} = 0 \) yields

\[
\frac{p_t}{d_t} = \frac{\beta}{1 - \beta}, \text{ or } \frac{p(z, d)}{d} = \frac{\beta}{1 - \beta} \quad (5)
\]

The price-dividend ratio is constant, verifying our assumption that \( x \) depends only on \( z \).

(d) Next, (2) and (4) imply

\[
q(z'|z) = \beta \Pr (z'|z) \frac{d}{\mu(z') d}
\]

because, in equilibrium \( c = d \Rightarrow \frac{c}{d} = \frac{d}{p} \). Hence,

\[
q(z'|z) = \begin{cases} 
\frac{\beta p}{\mu + \sigma} & \text{if } z' = 1, z = 0 \\
\frac{\beta (1 - p)}{\mu + \sigma} & \text{if } z' = 1, z = 1
\end{cases}
\]

Similarly,

\[
q(z'|z) = \begin{cases} 
\frac{\beta p}{\mu - \sigma} & \text{if } z' = 0, z = 0 \\
\frac{\beta (1 - p)}{\mu - \sigma} & \text{if } z' = 0, z = 1
\end{cases}
\]

(e) The key is that the bond is a redundant asset. We can therefore determine the price, \( Q(z) \), simply as the sum of two assets that replicate the bond:

\[
Q(z) = q(z'|z) + q(z'|z') = q(z' = 1|z) + q(z' = 0|z), \quad (6)
\]

which implies that

\[
Q(z = 0) = \frac{\beta p}{\mu + \sigma} + \frac{\beta (1 - p)}{\mu - \sigma};
\]

\[
Q(z = 1) = \frac{\beta (1 - p)}{\mu + \sigma} + \frac{\beta p}{\mu - \sigma}
\]
(f) \( p = 0.5 \). Then \( Q(z = 0) = Q(z = 1) = \frac{\beta \mu}{\mu^2 - \sigma^2} = Q \) (say). Denote the average gross and net rate of returns on bond, as \( R^b \) and \( r^b \), respectively. Then using (6)

\[
\begin{align*}
R^b &= \frac{1}{Q} \mu^2 - \sigma^2; \\
r^b &= R^b - 1
\end{align*}
\]

Denote the average gross and net rate of returns on trees, as \( R^s \) and \( r^s \), respectively. Then

\[
\begin{align*}
R^s &= E_t \left( \frac{p_{t+1} + d_{t+1}}{p_t} \right) \\
&= E_t \left( \frac{\frac{\mu}{1-\beta} d_{t+1} + d_{t+1}}{1-\beta} d_t \right) \\
&= \frac{1}{\beta} E_t \left( \frac{d_{t+1}}{d_t} \right) \\
&= \frac{\mu}{\beta} \\
r^s &= R^s - 1
\end{align*}
\]

The equity premium

\[
\gamma = R^s - R^b = \frac{\mu}{\beta} - \frac{\mu^2 - \sigma^2}{\beta \mu} = \frac{\sigma^2}{\beta \mu}
\]