Review Questions: Asset Pricing

Econ720. Fall 2021. Prof. Lutz Hendricks

---

1 Many Trees

Consider the standard Lucas fruit tree model. Assume that household i initially holds K_i trees of type i. Each tree produces a dividend d_{it} that is independently drawn from some distribution $G(d)$.

1. Define a competitive equilibrium.
2. Characterize the optimal portfolio held by each household.
3. Show that every household enjoys constant consumption, even without having access to Arrow securities.

Note: The answer is kind of obvious. I have not formally derived it. So I don't know how hard that is.

2 Lucas Trees

Consider a version of Lucas's asset pricing model with one representative household who is endowed with one tree.

Technology: The tree produces a stream of dividends, d_t, where $d_0 = 1$. The dividend growth rate, $\frac{d_{t+1}}{d_t}$, can take on one of two values, $\mu + \sigma$ or $\mu - \sigma$, where $\mu > 1$. The dividend growth rate is a Markov chain with transition matrix P. In particular, assume P is a symmetric matrix where the probability of *switching* growth rates is p, where $p \in (0, 1)$.

Preferences: Household preferences are given by $E \sum_{t=0}^{\infty} \beta^t \ln(c_t)$.

Markets: At each date, there are markets for consumption goods, trees, and state-contingent claims that pay one unit of consumption tomorrow in a particular state of the world.

Questions: (a) Define a solution to the household problem. Think carefully about what the household's state variables are.

(b) Define a recursive competitive equilibrium.

(c) Solve for the equilibrium pricing function for trees. You should find that the price-dividend ratio for trees is constant over time.

(d) Solve for the pricing functions for state-contingent claims. (Assume $\lim_{n\to\infty} \beta^{n+1} E_t x_{t+n} = 0$ to rule out bubbles).

(e) Add a riskless bond to this economy (a sure claim to one unit of the consumption good next period). Compute the price of a riskless bond. Hint: There is no need to resolve for the equilibrium price functions (why not?).

(f) Now assume $p = 0.5$. Compute the *average* rate of return on bonds and trees. What is the equity premium for this economy?

2.1 Answer: Lucas Trees

(a) Let $\mu(z) = \frac{d}{d_{-1}}$. The household's state variables are the exogenous states (z, d) and last period's choices $s, y(0), y(1)$. The controls are $s', y'(0), y'(1)$. The Bellman equation is

$$v(s, y(0), y(1), z, d) = \max_{s', y'(1), y'(0)} \{\ln c + \beta E[v(s', y'(0), y'(1), z', d')]\}$$

subject to the budget constraint

$$\begin{aligned} c + p(z,d)\, s' + y'(1)\, q(1|z) + y'(0)\, q(0|z) &= s(p(z,d) + d) + y(z) \\ d' &= \mu(z')\, d \end{aligned}$$

More explicitly

$$\begin{aligned} v(s, y(0), y(1), z, d) &= \max u\left(s(p(z,d) + d) + y(z) - p(z,d)\, s' - \sum_{z'} y'(z')\, q(z'|z)\right) \\ &\quad + \beta \sum_{z'} \Pr(z'|z)\;\; v(s', y'(0), y'(1), z', d\mu(z')) \end{aligned}$$

FOCs:

$$p(z,d)\;\; u'(c) = \beta \sum_{z'} \Pr(z'|z)\;\; v_1(s', y'(0), y'(1), z', \mu(z')\, d) \tag{1}$$

$$q(z'|z)\;\; u'(c) = \beta \Pr(z'|z)\;\; v_{y'(z')}(s', y'(0), y'(1), z', \mu(z')\, d) \tag{2}$$

where the latter holds for all z'. Envelope conditions:

$$v_1(.) = u'(c)(p(z,d) + d) \tag{3}$$

$$v_{y(z)}(.) = u'(c) \tag{4}$$

The Euler equation is standard:

$$u'(c)\;\; p(z,d) = \beta \sum_{z'} \Pr(z'|z)\;\; u'(c')\;\; [p(z', d\mu(z')) + d\mu(z')]$$

There is another Euler equation for state-contingent claims:

$$u'(c)\;\; q(z'|z) = \beta\;\; \Pr(z'|z)\;\; u'(c')$$

where it is understood that c' is the realization in the right state tomorrow.

A solution to the household problem consists of a value function and policy functions for $s'(s, y(0), y(1), z, d)$ and $y'(z'; s, y(0), y(1), z, d)$ such that:

- v satisfies the fixed-point property of the Bellman equation, given optimal policies.
- Optimal policies maximize the right hand side of the Bellman equation, given v.

(b) A recursive competitive equilibrium is:

1. A set of individual decision rules,
2. A set of pricing functions $p(z,d), q(z'|z)$,

such that:

1. Given pricing functions the decision rules solve the household's dynamic program (see (a)).
2. Markets clear:

$$\begin{aligned} s'(1, 0, 0, z, d) &= 1 \\ y'(z'; z, 0, 0, z, d) &= 0 \end{aligned}$$

(c) It's convenient to use time subscripts now:

$$\frac{p_t}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left(p_{t+1} + d_{t+1} \right) \right]$$

In equilibrium $c_t = d_t$. Hence,

$$\frac{p_t}{d_t} = \beta E_t \left[\frac{1}{d_{t+1}} \left(p_{t+1} + d_{t+1} \right) \right]$$

Define $x = p/d$. Iterating forward and using law of iterated expectations then yields

$$\begin{aligned} x_t &= \beta + \beta^2 + \beta^2 E_t \left[E_{t+1} \left\{ x_{t+1} \right\} \right] \\ &= \beta + \beta^2 + \beta^3 + ... + \beta^{n+1} + \beta^{n+1} E_t E_{t+1} ... E_{n-1} \left[E_n \left\{ x_n \right\} \right] \\ &= \beta + \beta^2 + \beta^3 + ... + \beta^{n+1} + \beta^{n+1} E_t x_{t+n} \end{aligned}$$

Assuming $\lim_{n \to \infty} \beta^{n+1} E_t x_{t+n} = 0$ yields

$$\begin{aligned} \frac{p_t}{d_t} &= \frac{\beta}{1-\beta}, \text{ or} \\ p(z,d) &= \frac{\beta}{1-\beta} d \end{aligned} \tag{5}$$

The price-dividend ratio is constant, verifying our assumption that x depends only on z.

(d) Next, (2) and (4) imply

$$q(z'|z) = \beta \ \Pr(z'|z) \frac{d}{\mu(z') \ d}$$

because, in equilibrium $c = d \Longrightarrow \frac{c}{c'} = \frac{d}{d'}$. Hence,

$$\begin{aligned} q(z' = 1|z = 0) &= \frac{\beta p}{\mu + \sigma}; \\ q(z' = 1|z = 1) &= \frac{\beta(1-p)}{\mu + \sigma}; \end{aligned}$$

Similarly,

$$\begin{aligned} q(z' = 0|z = 0) &= \frac{\beta p}{\mu - \sigma}; \\ q(z' = 0|z = 1) &= \frac{\beta(1-p)}{\mu - \sigma}; \end{aligned}$$

(e) The key is that the bond is a redundant asset. We can therefore determine the price, $Q(z)$, simply as the sum of two assets that replicate the bond:

$$Q(z) = q(z' = 1|z) + q(z' = 0|z), \tag{6}$$

which implies that

$$\begin{aligned} Q(z = 0) &= \frac{\beta p}{\mu + \sigma} + \frac{\beta(1-p)}{\mu - \sigma}; \\ Q(z = 1) &= \frac{\beta(1-p)}{\mu + \sigma} + \frac{\beta p}{\mu - \sigma} \end{aligned}$$

(f) $p = 0.5$. Then $Q(z=0) = Q(z=1) = \frac{\beta\mu}{\mu^2-\sigma^2} = Q$ (say). Denote the average gross and net rate of returns on bond, as R^b and r^b, respectively. Then using (6)

$$\begin{aligned} R^b &= \frac{1}{Q} \\ &= \frac{\mu^2-\sigma^2}{\beta\mu}; \\ r^b &= R^b - 1 \end{aligned}$$

Denote the average gross and net rate of returns on trees, as R^s and r^s, respectively. Then

$$\begin{aligned} R^s &= E_t\left(\frac{p_{t+1}+d_{t+1}}{p_t}\right) \\ &= E_t\left(\frac{\frac{\beta}{1-\beta}d_{t+1}+d_{t+1}}{\frac{\beta}{1-\beta}d_t}\right) \\ &= \frac{1}{\beta}E_t\left(\frac{d_{t+1}}{d_t}\right) \\ &= \frac{\mu}{\beta} \\ r^s &= R^s - 1 \end{aligned}$$

The equity premium

$$\begin{aligned} \gamma &= R^s - R^b \\ &= \frac{\mu}{\beta} - \frac{\mu^2-\sigma^2}{\beta\mu} \\ &= \frac{\sigma^2}{\beta\mu} \end{aligned}$$