1 Lucas Fruit Trees With Crashes

Demographics: There is a single, representative household who lives forever.
Preferences: \( U = E \sum_{t=0}^{\infty} \beta^t u(c_t) \) where \( u(c) = c^{1-\sigma} / (1 - \sigma) \).
Endowments: The agent is endowed at \( t = 0 \) with 1 tree. In each period, the tree yields stochastic consumption \( d_t \), which cannot be stored. \( d_t \) evolves as follows:

- If \( d_t = d_{t-1} \), then \( d_{t+1} = d_t \) forever after.
- If \( d_t \neq d_{t-1} \), then \( d_{t+1} = \gamma d_t \) with probability \( \pi \) and \( d_{t+1} = d_t \) with probability \( 1 - \pi \). \( \gamma > 1 \).

In words: \( d \) grows at rate \( \gamma - 1 \) until some random event occurs (with probability \( 1 - \pi \)), at which point growth stops forever.
Markets: There are competitive markets for consumption (numeraire) and trees (price \( p_t \)). Assume that \( p_t \) is \emph{cum dividend}, meaning that \( d_t \) accrues to the household who buys the tree in \( t \) and holds it into \( t + 1 \).

Questions:

1. State the household’s dynamic program.
2. Derive the Euler equation.
3. Define a recursive competitive equilibrium. Key: what is the state vector?
4. Characterize the stochastic process of \( p_t \). Is \( p_t \) a Markov process? Hint: there are 2 phases: before and after dividends have stopped growing. Assume that \( p/d \) is constant during the phase with growth.
5. What happens to the stock market when the economy stops growing? Does it crash? Under what condition?

2 Answer: Lucas Fruit Trees With Crashes

[Based on a question due to Rodolfo Manuelli]

1. The household problem is standard, except for the stock price being \emph{cum dividend}:

\[
V(k, S) = \max u(c) + \beta E \{ V(k', S') \} 
\]

where \( S \) is the aggregate state, subject to

\[
p(k' - k) + c = dk' \]

(1)
2. Euler:

\[ p_t = d_t + \beta E \left\{ \frac{u'(c_{t+1})}{u'(c_t)} p_{t+1} \right\} \]  \hspace{1cm} (3)

3. The distribution of \( d_{t+1} \) depends on \( d_t \) and \( d_{t-1} \). These form the state vector.

Objects: \( V(k; S), \kappa(k, S), p(S) \)

Equilibrium conditions:

- \( V \) and \( \kappa \) solve the household problem
- goods market clearing: \( c(k, S) = d \)
- asset market clearing: \( \kappa(k, S) = 1 \)
- law of motion of the aggregate state (exogenous)

4. Stock prices:

Once growth has stopped: \( c = d' = d \), so that \( p - d = \beta p \implies p = d / (1 - \beta) \).

While growth continues:

\[ p_g = d + \beta \pi \gamma p' + \beta (1 - \pi) d / (1 - \beta) \]  \hspace{1cm} (4)

Divide by \( d \):

\[ p_g / d = 1 + \beta \pi \gamma 1 - \sigma p_g' / d' + \beta (1 - \pi) / (1 - \beta) \]  \hspace{1cm} (5)

or:

\[ p_g / d = \frac{1 + \beta (1 - \pi) / (1 - \beta)}{1 - \beta \pi \gamma 1 - \sigma} \]  \hspace{1cm} (6)

The price is Markov in the state \( S = (d, d_{-1}) \).

5. The growth slowdown leads to a stock market crash, if \( p_g / d < p / d \) or

\[ \frac{1 + \beta (1 - \pi) / (1 - \beta)}{1 - \beta \pi \gamma 1 - \sigma} < \frac{1}{1 - \beta} \]  \hspace{1cm} (7)

or

\[ \beta \pi > \beta \pi \gamma 1 - \sigma \]  \hspace{1cm} (8)

Whether the growth slowdown crashes the stock market depends on the sign of \( 1 - \sigma \) (the curvature of preferences or the relative strengths of income and substitution effects).