Asset Pricing

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Topics

1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?

We use the Lucas (1978) fruit tree model.

- The implications are far more general than the simple model.
- The model forms the basis for the CAPM and the β risk measure.

The Lucas (1978) Fruit Tree Model

- **Agents**:
  - A single representative household.
- **Preferences**:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

- E_0 is the expectation as of time $t = 0$.

Technology

- This is an endowment economy.
- There are K identical fruit trees.
- Each tree yields d_t units of consumption goods in period t.
- d_t is random and the same for all trees.
- Trees cannot be produced.
- Fruits cannot be stored.

Technology

- The aggregate resource constraint:

$$c_t = Kd_t \tag{2}$$

- Assume that d is a finite Markov chain with transition matrix $\pi(d', d)$.
- An important feature: All uncertainty is **aggregate**.
- There are no opportunities for households to insure each other.
- This is why we can work with a representative household.

Markets

- There are markets for fruits and for trees.
- There is also a one period bond, issued by households (in zero net supply).
  - Its purpose is to determine a risk-free interest rate.
- Digression: Why this is a good model...

Household problem

- The household starts out with bonds (b_0) and shares (k_0).
- At each date, he chooses c_t, b_{t+1}, k_{t+1}.
- The **budget constraint** is

$$p_t k_{t+1} + b_{t+1} = R_t b_t + (p_t + d_t) k_t - c_t \tag{3}$$

- Notation:
  - p : the price of trees. Suppressing dependence on the state.
  - R : the real interest rate on bonds.
  - the price of bonds is normalized to 1 (how?).

Household problem

$$V(k,b,d) = \max u(c) + \beta EV\left(k',b',d'\right) \tag{4}$$

subject to

$$Rb + (p+d)k - c + pk' - b' = 0 \tag{5}$$

Household problem

First-order conditions:

$$\begin{aligned} c &: \quad u'(c) = \lambda \\ k' &: \quad \lambda p = EV_k(k', b', d') \\ b' &: \quad \lambda = EV_b\left(k', b', d'\right) \end{aligned}$$

Envelope:

$$\begin{aligned} V_k &= \lambda (p + d) \\ V_b &= \lambda R \end{aligned}$$

Euler equations

$$
\begin{aligned}
u'(c_t) &= \beta E_t \left\{ u'(c_{t+1}) R_{t+1} \right\} \\
&= \beta E_t \{ u'(c_{t+1}) \underbrace{\frac{p_{t+1} + d_{t+1}}{p_t}}_{R^S_{t+1}} \}
\end{aligned}
$$

This is very general - holds for any number of assets / for any type of asset.

Question: Why doesn't the correlation of asset returns show up anywhere?

Solution

- A solution consists of state contingent plans $\{c(d^t), k(d^t), b(d^t)\}$ for all histories d^t.
- These satisfy:
  - 2 Euler equations
  - 1 budget constraint.
  - b_0 and k_0 given.
  - Transversality: $\lim_{t\to\infty} E_0 \beta^t u'(c_t)\ [b_t + p_t k_t] = 0$.

Market clearing

For every history we need:

Bonds:

$$b_t = 0$$

Trees:

$$k_t = K_t$$

Goods:

$$c_t = K_t d_t$$

There is no trade in equilibrium!

Competitive Equilibrium

- A CE consists of:
  1. an allocation: $\{c(d^t), b(d^t), k(d^t)\}$.
  2. a price system: $\{p(d^t), R(d^t)\}$
- These satisfy:
  1. household: 2 Euler equations and 1 budget constraint.
  2. 3 market clearing conditions.

Recursive Competitive Equilibrium

Objects:

- Solution to the household problem: $V(k,b,d)$ and $c(k,b,d)$, $k' = \kappa(k,b,d)$, $b' = B(k,b,d)$.
- Price functions: $p(d), R(d)$.

Equilibrium conditions:

- Household: 4
- Market clearing: 3
- No need for consistency: law of motion of the aggregate state is exogenous.

Consumption smoothing

The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \tag{6}$$

Define: Marginal rate of substitution:

$$MRS_{t+1} = \beta\, u'(c_{t+1})/u'(c_t) \tag{7}$$

MRS_{t+1} is inversely related to consumption growth.

Consumption smoothing

With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$:

$$\begin{aligned} u'(c) &= c^{-\sigma} && (8) \\ MRS_{t+1} &= \beta \ (c_{t+1}/c_t)^{-\sigma} && (9) \end{aligned}$$

Euler equation:

$$\beta \mathbb{E}_t \left\{ (1 + g(c_{t+1}))^{-\sigma} R_{t+1} \right\} = 1 \qquad (10)$$

High interest rate

$\Longrightarrow$ reward for postponing consumption

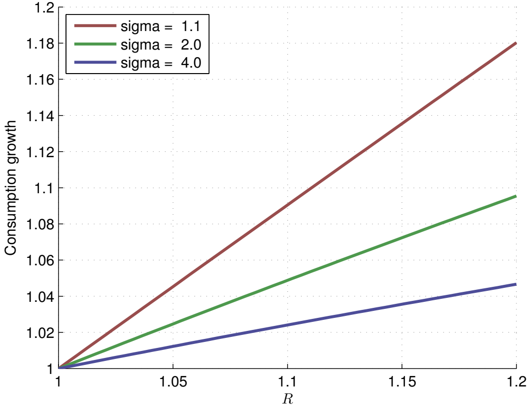
$\Longrightarrow$ high consumption growth

The **coefficient of relative risk aversion** (σ) determines how much consumption growth responds to interest rates.

Consumption smoothing

High σ implies that the household chooses smooth consumption.

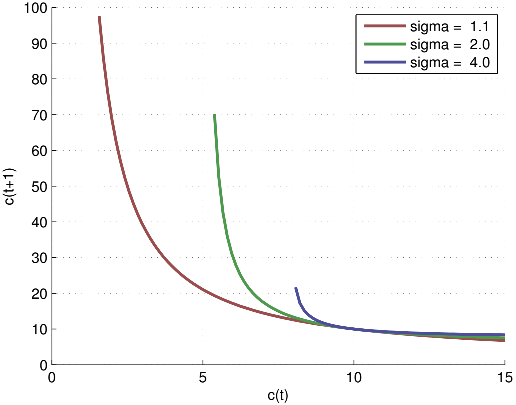
Illustration for the deterministic case:



Consumption smoothing

With high σ, marginal utility changes a lot when c changes.

The household then keeps c smooth.



Asset Prices

Asset pricing implications

- We will now derive the famous **Lucas asset pricing equation**.
- Define: Rate of return on trees: $R_{t+1}^{S} = (p_{t+1} + d_{t+1})/p_t$.
- Directly from the 2 Euler equations:

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1}^{S} \right\} = 1$$

- Or

$$E\{MRS_{t+1} R_{t+1}\} = E\left\{MRS_{t+1} R_{t+1}^{S}\right\} = 1 \tag{11}$$

When does an asset pay a high expected return?

Re-write asset pricing equation using

$$Cov(x,y) = E(xy) - E(x)E(y)$$

as

$$\begin{aligned} 1 &= E\{MRS\}\,E\{R\} + Cov(MRS,R) && (12) \\ E(R) &= \frac{1 - Cov(MRS,R)}{E(MRS)} && (13) \end{aligned}$$

When do assets pay high returns?

$$\mathbb{E}(R) = \frac{1 - Cov(MRS, R)}{\mathbb{E}(MRS)} \tag{14}$$

- Take a "safe" asset with fixed R.
  - $Cov(MRS, R) = 0$
  - $\mathbb{E}(R) = 1/\mathbb{E}(MRS)$.
- If $Cov(MRS, R) < 0$: the asset pays higher return than the safe asset
  - a **risk premium**
- If $Cov(MRS, R) > 0$: the asset pays **lower** return than the safe asset
  - important point: an asset return can have lots of volatility, but pay a lower return than a t-bill
  - examples?

When do assets pay high returns?

High returns require low / negative $Cov(MRS, R)$.

Example: log utility

- $u'(c) = 1/c$
- $MRS = \beta u'(c_{t+1})/u'(c_t) = \beta c_t / c_{t+1}$.

High MRS means low consumption growth.

Key implication

Assets are **risky** if their returns are **positively correlated with consumption growth**.
Then they have high **expected returns**.

Note: Uncertainty about R by itself is not priced.

- Only the part that is correlated with consumption growth is priced.

Intuition

- Imagine there are good times (high c) and bad times (low c).
- There are 2 assets: A pays dividends in good times, B pays in bad times.
- The value of the dividend is $u'(c)$.
- Assets that pay in good times are not valuable: $u'(c)$ is low.
- Assets that pay in bad times provide insurance - they are valuable (have low expected returns).

Risk (premia)

- The "risk free" assets has expected return

$$E(R_f) = \frac{1}{E(MRS)} \tag{15}$$

- A "risky" asset has expected return

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \tag{16}$$

- The **risk premium** is

$$E(R) - E(R_f) = -\frac{Cov(MRS, R)}{E(MRS)} \tag{17}$$

- This defines what **risk** means: **covariance with consumption growth**.
- Note that risk can be **negative** (insurance).

The Equity Premium Puzzle

- Mehra and Prescott (1985): Asset return data pose a puzzle for the theory.
- The equity premium is "high" (6-7% p.a.)
- The cov of c growth and R_s is low.
  - The reason: Consumption is very smooth.

The Equity Premium Puzzle

TABLE 1
SUMMARY STATISTICS
UNITED STATES ANNUAL DATA, 1889–1978

	Sample Means		
R_t^s	0.070		
R_t^b	0.010		
C_t/C_{t-1}	0.018		
	Sample Variance-Covariance		
	R_t^s	R_t^b	C_t/C_{t-1}
R_t^s	0.0274	0.00104	0.00219
R_t^b	0.00104	0.00308	–0.000193
C_t/C_{t-1}	0.00219	–0.000193	0.00127

The Equity Premium Puzzle

A back-of-the envelope calculation with CRRA utility:

$$EP = -\frac{Cov\left(\beta\left[c_{t+1}/c_t\right]^{-\sigma},\ R_s\right)}{E\left\{\beta\left[c_{t+1}/c_t\right]^{-\sigma}\right\}} \tag{18}$$

Take log utility: $\sigma = 1$.

- $Cov(MRS, R_s) \simeq -0.0022$.
- $E(MRS) \simeq 1$.
- $EP \simeq 0.2\%$.
- Replicating the observed equity premium requires **very high risk aversion** $(\sigma = 40)$.

How severe is the puzzle?

Investors forego very large returns.

Table 3

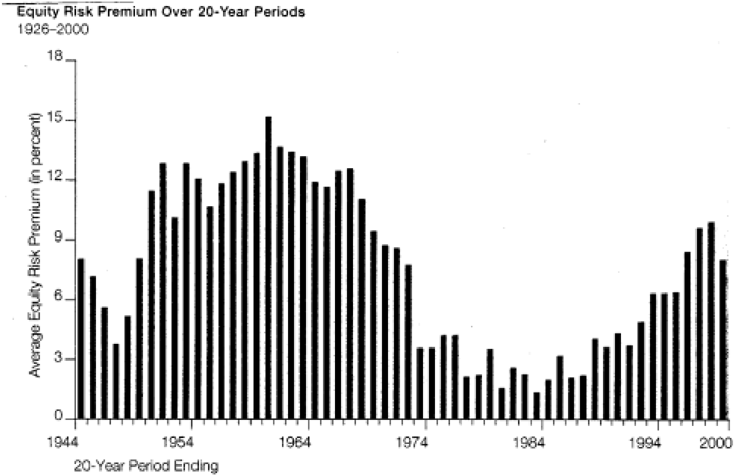
Terminal value of $1 invested in Stocks and Bonds

Investment Period	**Stocks**		**T-bills**	
	Real	Nominal	Real	Nominal
1802-1997	$558,945	$7,470,000	$276	$3,679
1926-2000	$266.47	$2,586.52	$1.71	$16.56

Source: Mehra and Prescott (2003)

Long holding periods

Over 20 year holding periods: stocks dominate bonds.



Source: Mehra and Prescott (2003)

Why do we care?

- The EP puzzle shows that we do not understand
  1. what households view as "risky"
  2. why households place a high value on smooth consumption
- This has implications for:
  1. The welfare costs of business cycles
     - They are very low in standard models.
  2. Stock price volatility.
     - Standard models fail to explain it (see below).

How to resolve the puzzle

Proposed explanations include:

1. Habit formation: $u(c_t, c_{t-1}) = \frac{[c_t - \gamma c_{t-1}]^{1-\sigma}}{1-\sigma}$.
   - Implies high risk aversion when c_t is close to c_{t-1}.
2. Heterogeneous agents
   - Implicit in the standard model: all idiosyncratic risk is perfectly insured.
3. Borrowing constraints
   - The young should hold stocks (long horizon), but cannot.
   - The old receive mostly capital income and find stocks risky.
4. Taxes / regulations (McGrattan and Prescott, 2000)
   - The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.

Beta

Now we derive the famous "beta" measure of risk.

Suppose asset m (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma R_{m,t+1} \tag{19}$$

The market's expected return is

$$E\,R_m - R = -\frac{Cov(MRS, R_m)}{E(MRS)} \tag{20}$$

Beta

Now we relate the covariance term to marginal utility:

$$\begin{aligned} Cov\left(MRS, R_m\right) &= Cov\left(\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R_{m,t+1}\right) = \beta \frac{Cov\left(u'\left(c_{t+1}\right), R_{m,t+1}\right)}{u'\left(c_t\right)} \\ E\left(MRS\right) &= \beta \frac{E\left(u'\left(c_{t+1}\right)\right)}{u'\left(c_t\right)} \end{aligned}$$

Therefore:

$$E\left(R_m\right) - R = -\frac{Cov\left(u'\left(c_{t+1}\right), R_{m,t+1}\right)}{E\ u'\left(c_{t+1}\right)} = \frac{\gamma\ Var\left(R_{m,t+1}\right)}{E\ u'\left(c_{t+1}\right)}$$

Beta

For any asset i:

$$E\ R_i - R = -\frac{Cov\left(u'\left(c_{t+1}\right), R_i\right)}{E\ u'\left(c_{t+1}\right)} = \frac{\gamma\ Cov\left(R_m, R_i\right)}{E\ u'\left(c_{t+1}\right)}$$

Take the ratio for assets i and m:

$$\beta_i = \frac{\mathbb{E}R_i - R}{\mathbb{E}R_m - R} = \frac{Cov\left(R_m, R_i\right)}{Var\left(R_m\right)} \tag{21}$$

Note: β_i is the coefficient of regressing R_i on R_m using OLS.

This is the famous **CAPM** asset pricing equation.

Beta

- The risk premium for asset i depends on:
  - it's **beta** (essentially the correlation with the market)
  - the market price of risk: $E\ R_m - R$.
- A stock's beta can be estimated from data on past returns of the stock (R_i) and the market (using a broad stock index).
- Betas are used to
  - Measure the risk of an asset.
  - Calculate the required rate of return for investment projects.
  - Evaluation of mutual fund managers.

Securities market line

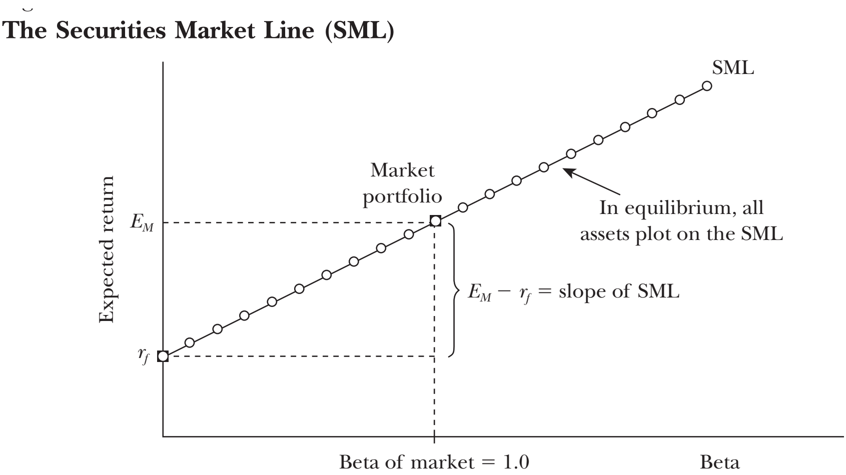
CAPM prediction:

$$\mathbb{E}R_i = (1-\beta_i)R + \beta_i \mathbb{E}R_m \tag{22}$$

$$= R + \beta_i \mathbb{E}\{R_m - R\} \tag{23}$$

If we plot expected returns against βs, we should get a straight line. This is called the **securities market line** (SML)

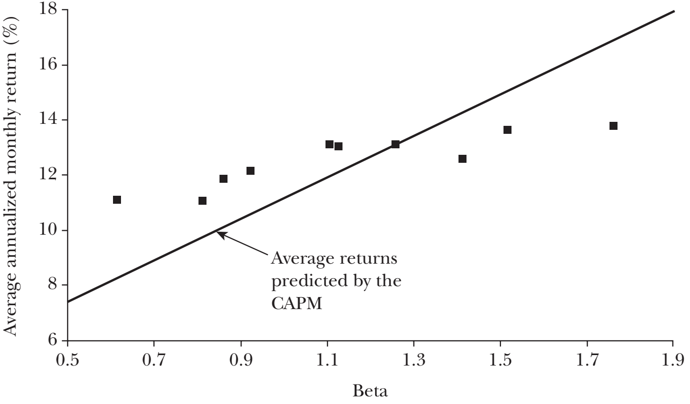
Securities market line



Source: Perold (2004)

Securities market line: Evidence

Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003



Source: Fama (2004)

Implications

Stocks with higher βs have higher expected returns, but the relationship is flatter than predicted.

Again: we don't understand how investors value / measure risk.

- a fundamental problem.

Oddly, β remains popular, even though it does not work in the data.

Solving for the asset price

We show that the asset price equals the present discounted value of dividends

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} d_{t+j} MRS(t, t+j) \tag{24}$$

The discount factor is the MRS, called the **stochastic discount factor**.

Solving for the asset price

Start from the Euler equation:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \tag{25}$$

Solve for the price:

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \tag{26}$$

Replace p_{t+1} with (26) shifted to $t+1$:

$$\begin{aligned} p_t \quad = \quad & E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\ & E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[\frac{\beta u'(c_{t+2})}{u'(c_{t+1})} \right] (p_{t+2} + d_{t+2}) \right\} \end{aligned} \tag{27}$$

Solving for the asset price

The law of iterated expectations:

$$E_t\left\{E_{t+1}(x)\right\} = E_t(x) \tag{28}$$

Eliminate the E_{t+1}:

$$p_t = E_t\left\{\frac{\beta\, u'(c_{t+1})}{u'(c_t)}\, d_{t+1}\right\} + E_t\left\{\frac{\beta^2\, u'(c_{t+2})}{u'(c_t)}\left(p_{t+2} + d_{t+2}\right)\right\} \tag{29}$$

Iterate forward for T periods:

$$p_t = E_t\left\{\sum_{j=1}^{T}\frac{\beta^j\, u'(c_{t+j})}{u'(c_t)}\, d_{t+j}\right\} \tag{30}$$

$$+E_t\left\{\frac{\beta^T u'(c_{t+T})}{u'(c_t)}\, p_{t+T}\right\} \tag{31}$$

Solving for the asset price

Impose that the last term vanishes in the limit:

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \tag{32}$$

- There is no good reason for this assumption!
- We will see later: other prices solve the asset pricing equation (bubbles)

The asset price equals the **discounted present value of dividends**.

The stochastic **discount factor** is the marginal rate of substitution.

Example: Log Utility

In the Lucas model, assume: $u(c) = \ln(c)$. $K = 1$.

In equilibrium: $c_t = d_t$.

$MRS_{t+1} = \frac{\beta \ u'(c_{t+1})}{u'(c_t)} = \frac{\beta \ d_t}{d_{t+1}}$.

The asset pricing equation becomes

$$
\begin{aligned}
p_t &= E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j d_t}{d_{t+j}} d_{t+j} \right\} \\
&= d_t \frac{\beta}{1-\beta}
\end{aligned}
$$

Example: Periodic dividends

In the Lucas model, assume:

- Utility is $u(c) = c^{1-\sigma}/(1-\sigma)$.
- d_t alternates between d^H and d^L.

Asset pricing equation:

$$\begin{aligned} p_t &= \sum \beta^j \left(d_t / d_{t+j}\right)^{\sigma} d_{t+j} \\ &= d_t^{\sigma} \sum \beta^j d_{t+j}^{1-\sigma} \end{aligned} \tag{33}$$

On good days, p_t is pulled up by low $u'(c')$, but is pushed down by low d_{t+1}.

The Excess Volatility Puzzle

Consider a stock with dividend process d_t.

Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \tag{34}$$

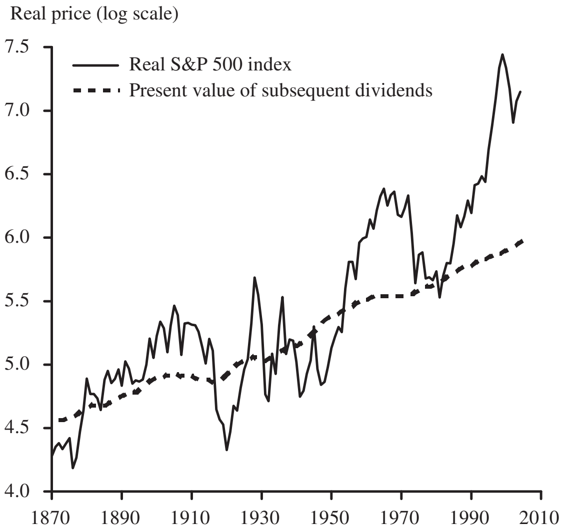
In the data:

- Dividends are very smooth (a goal of company policy).
- Stock prices are much more volatile than dividends.

But in the theory: stock prices should be the **average** of future dividends and thus **smoother** than dividends.

This is the flip-side of the Equity Premium Puzzle. See Shiller (1981)

Excess Volatility



Source: FRSBSF Economic Letter Nov 2007

Bubbles

- Recall how the asset pricing formula is derived:
- We iterate forward on the asset pricing Euler equation

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \tag{35}$$

- We assume that the p_{t+1} term vanishes in the limit.
- What if it does not vanish?
- Then **any** (current) **asset price** can satisfy the asset pricing equation.
- The deviation between p_t and the **fundamental price** from (35) is called a **bubble**.
- It is purely a self-fulfilling expectation.

Bubbles: Example

- Consider an asset that pays no dividends.
- Its **fundamental price** is 0.
- Assume that the MRS is constant at $\frac{\beta \ u'(c_{t+1})}{u'(c_1)} = 1$.
- The the asset pricing equation is

$$p_t = E_t p_{t+1} \tag{36}$$

- One price process that satisfies this: p doubles with probability 1/2 and drops to 0 otherwise.
- This satisfies (36) for **any** p_t.
- Bubbles are a possible explanation for asset price volatility.
- Note that the bubble does not offer any excess return opportunities.

Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.
- On the equity premium puzzle: Mehra and Prescott (1985, 2003)

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