1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?

- We use the Lucas (1978) fruit tree model.
- The implications are far more general than the simple model.
- The model forms the basis for the CAPM and the $\beta$ risk measure.
The Lucas (1978) Fruit Tree Model

▶ Agents:
  ▶ A single representative household.

▶ Preferences:

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}
\]

▶ \(E_0\) is the expectation as of time \(t = 0\).
This is an endowment economy.

There are $K$ identical fruit trees.

Each tree yields $d_t$ units of consumption goods in period $t$.

$d_t$ is random and the same for all trees.

Trees cannot be produced.

Fruits cannot be stored.
Technology

- The aggregate resource constraint:

\[ c_t = Kd_t \]  

(2)

- Assume that \( d \) is a finite Markov chain with transition matrix \( \pi(d',d) \).

- An important feature: All uncertainty is aggregate.

- There are no opportunities for households to insure each other.

- This is why we can work with a representative household.
Markets

- There are markets for fruits and for trees.
- There is also a one period bond, issued by households (in zero net supply).
  - Its purpose is to determine a risk-free interest rate.
Household problem

- The household starts out with bonds \((b_0)\) and shares \((k_0)\).
- At each date, he chooses \(c_t, b_{t+1}, k_{t+1}\).
- The budget constraint is

\[
p_t k_{t+1} + b_{t+1} = R_t b_t + (p_t + d_t) k_t - c_t
\]  

(3)

- Notation:
  - \(p\): the price of trees. Suppressing dependence on the state.
  - \(R\): the real interest rate on bonds.
  - the price of bonds is normalized to 1 (how?).
Household problem

\[ V(k, b, d) = \max u(c) + \beta EV(k', b', d') \]  \hspace{1cm} (4)

subject to

\[ Rb + (p + d)k - c + pk' - b' = 0 \]  \hspace{1cm} (5)
Household problem

First-order conditions:

\[
\begin{align*}
\quad c & : \quad u'(c) = \lambda \\
\quad k' & : \quad \lambda p = EV_k(k', b', d') \\
\quad b' & : \quad \lambda = EV_b(k', b', d')
\end{align*}
\]

Envelope:

\[
\begin{align*}
V_k & = \lambda (p + d) \\
V_b & = \lambda R
\end{align*}
\]
Euler equations

\[ u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) R_{t+1} \right\} \]

\[ = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \]

This is very general - holds for any number of assets / for any type of asset.
Solution

- A solution consists of state contingent plans \( \{c(d^t), k(d^t), b(d^t)\} \) for all histories \( d^t \).

- These satisfy:
  - 2 Euler equations
  - 1 budget constraint.
  - \( b_0 \) and \( k_0 \) given.
  - Transversality: \( \lim_{t \to \infty} E_0 \beta^t u'(c_t) [b_t + p_t k_t] = 0 \).
Market clearing

For every history we need:

Bonds:

\[ b_t = 0 \]

Trees:

\[ k_t = K_t \]

Goods:

\[ c_t = K_t d_t \]

There is no trade in equilibrium!
Competitive Equilibrium

A CE consists of:
1. an allocation: \( \{c(d^l), b(d^l), k(d^l)\} \).
2. a price system: \( \{p(d^l), R(d^l)\} \)

These satisfy:
1. household: 2 Euler equations and 1 budget constraint.
2. 3 market clearing conditions.
Recursive Competitive Equilibrium

Objects:

- Solution to the household problem: $V(k, b, d)$ and $c(k, b, d)$, $k' = \kappa(k, b, d)$, $b' = B(k, b, d)$.
- Price functions: $p(d), R(d)$.

Equilibrium conditions:

- Household: 4
- Market clearing: 2
- No need for consistency: law of motion of the aggregate state is exogenous.
Consumption smoothing

- The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \quad (6)$$

- Define: Marginal rate of substitution:

$$MRS_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} \quad (7)$$

- $MRS_{t+1}$ is inversely related to consumption growth.

- With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$:

$$u'(c) = c^{-\sigma} \quad (8)$$

$$MRS_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \quad (9)$$
Consumption smoothing

- The coefficient of relative risk aversion ($\sigma$) determines how much $MRS$ fluctuates with $c$.
- High $\sigma$ implies that the household chooses smooth consumption.
- Illustration for the deterministic case:
Consumption smoothing

- With high $\sigma$, marginal utility changes a lot when $c$ changes.
- The household then keeps $c$ smooth.
Asset Prices
Asset pricing implications

- We will now derive the famous **Lucas asset pricing equation**.
- Define: Rate of return on trees: \( R_{t+1}^S = \left( p_{t+1} + d_{t+1} \right) / p_t \).
- Directly from the 2 Euler equations:

\[
E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1}^S \right\} = 1
\]

- Or

\[
E \{ \text{MRS}_{t+1} R_{t+1} \} = E \{ \text{MRS}_{t+1} R_{t+1}^S \} = 1 \quad (10)
\]
When does an asset pay a high expected return?

Re-write asset pricing equation using

\[ \text{Cov}(x, y) = E(xy) - E(x)E(y) \]

as

\[ 1 = E\{MRS\}E\{R\} + \text{Cov}(MRS, R) \] (11)

\[ E(R) = \frac{1 - \text{Cov}(MRS, R)}{E(MRS)} \] (12)
When do assets pay high returns?

\[ \mathbb{E}(R) = \frac{1 - \text{Cov}(MRS, R)}{\mathbb{E}(MRS)} \]  \hspace{1cm} (13)

- Take a “safe” asset with fixed $R$.
  - \( \text{Cov}(MRS, R) = 0 \)
  - \( \mathbb{E}(R) = 1/\mathbb{E}(MRS) \).
- If \( \text{Cov}(MRS, R) < 0 \): the asset pays higher return than the safe asset
  - a risk premium
- If \( \text{Cov}(MRS, R) > 0 \): the asset pays lower return than the safe asset
  - important point: an asset return can have lots of volatility, but pay a lower return than a t-bill
  - examples?
When do assets pay high returns?

- High returns require low / negative $\text{Cov}(\text{MRS}, R)$.
- Example: log utility
  - $u'(c) = 1/c$
  - $\text{MRS} = \beta u'(c_{t+1})/u'(c_t) = \beta c_t/c_{t+1}$.
- High $\text{MRS}$ means low consumption growth.
- Therefore: Assets pay high returns if their returns are positively correlated with consumption growth.
Imagine there are good times (high $c$) and bad times (low $c$).
There are 2 assets: A pays dividends in good times, B pays in bad times.
The value of the dividend is $u'(c)$.
Assets that pay in good times are not valuable: $u'(c)$ is low.
Assets that pay in bad times provide insurance - they are valuable (have low expected returns).
Risk (premia)

- The "risk free" assets has expected return

\[ E(R_f) = \frac{1}{E(MRS)} \]  

(14)

- A "risky" asset has expected return

\[ E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \]  

(15)

- The risk premium is

\[ E(R) - E(R_f) = -\frac{Cov(MRS, R)}{E(MRS)} \]  

(16)

- This defines what risk means: covariance with consumption growth.
- Note that risk can be negative (insurance).
The Equity Premium Puzzle

- The equity premium is "high" (6-7% p.a.)
- The cov of $c$ growth and $R_s$ is low.
  - The reason: Consumption is very smooth.
TABLE 1
SUMMARY STATISTICS
UNITED STATES ANNUAL DATA, 1889–1978

<table>
<thead>
<tr>
<th>Sample Means</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^s_t$</td>
<td>0.070</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^b_t$</td>
<td>0.010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_t/C_{t-1}$</td>
<td>0.018</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Variance-Covariance</th>
<th>$R^s_t$</th>
<th>$R^b_t$</th>
<th>$C_t/C_{t-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^s_t$</td>
<td>0.0274</td>
<td>0.00104</td>
<td>0.00219</td>
</tr>
<tr>
<td>$R^b_t$</td>
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<td>0.00308</td>
<td>−0.000193</td>
</tr>
<tr>
<td>$C_t/C_{t-1}$</td>
<td>0.00219</td>
<td>−0.000193</td>
<td>0.00127</td>
</tr>
</tbody>
</table>
The Equity Premium Puzzle

A back-of-the-envelope calculation with CRRA utility:

\[
EP = - \frac{\text{Cov}(\beta \frac{c_{t+1}}{c_t}^{-\sigma}, R_s)}{E\{\beta \frac{c_{t+1}}{c_t}^{-\sigma}\}}
\]  

(17)

Take log utility: \( \sigma = 1 \).

\begin{itemize}
  \item \( \text{Cov}(MRS, R_s) \approx -0.0022 \).
  \item \( E(MRS) \approx 1 \).
  \item \( EP \approx 0.2\% \).
  \item Replicating the observed equity premium requires very high risk aversion (\( \sigma = 40 \)).
\end{itemize}
How severe is the puzzle?

Investors forego very large returns.

Table 3

<table>
<thead>
<tr>
<th>Investment Period</th>
<th>Stocks</th>
<th>T-bills</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real</td>
<td>Nominal</td>
</tr>
<tr>
<td>1802-1997</td>
<td>$558,945</td>
<td>$7,470,000</td>
</tr>
<tr>
<td>1926-2000</td>
<td>$266.47</td>
<td>$2,586.52</td>
</tr>
</tbody>
</table>

Long holding periods

Over 20 year holding periods: stocks dominate bonds.

Why do we care?

- The EP puzzle shows that we do not understand
  1. what households view as "risky"
  2. why households place a high value on smooth consumption

- This has implications for:
  1. The welfare costs of business cycles
    - They are very low in standard models.
  2. Stock price volatility.
    - Standard models fail to explain it (see below).
How to resolve the puzzle

Proposed explanations include:

1. Habit formation: $u(c_t, c_{t-1}) = \frac{[c_t - \gamma c_{t-1}]^{1-\sigma}}{1-\sigma}$.
   ▶ Implies high risk aversion when $c_t$ is close to $c_{t-1}$.

2. Heterogeneous agents
   ▶ Implicit in the standard model: all idiosyncratic risk is perfectly insured.

3. Borrowing constraints
   ▶ The young should hold stocks (long horizon), but cannot.
   ▶ The old receive mostly capital income and find stocks risky.

4. Taxes / regulations (McGrattan and Prescott, 2000)
   ▶ The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.
Now we derive the famous "beta" measure of risk. Suppose asset $m$ (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma R_{m,t+1} \quad (18)$$

The market’s expected return is

$$E R_m - R = -\frac{\text{Cov}(MRS, R_m)}{E(MRS)} \quad (19)$$
Now we relate the covariance term to marginal utility:

\[
\text{Cov}(MRS, R_m) = \text{Cov}\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}, R_{m,t+1}\right) = \beta \frac{\text{Cov}(u'(c_{t+1}), R_{m,t+1})}{u'(c_t)}
\]

\[
E(MRS) = \beta \frac{E(u'(c_{t+1}))}{u'(c_t)}
\]

Therefore:

\[
E(R_m) - R = -\frac{\text{Cov}(u'(c_{t+1}), R_{m,t+1})}{E u'(c_{t+1})} = \gamma \frac{\text{Var}(R_{m,t+1})}{E u'(c_{t+1})}
\]
For any asset $i$:

$$E R_i - R = - \frac{\text{Cov}(u'(c_{t+1}), R_i)}{E u'(c_{t+1})} = \gamma \frac{\text{Cov}(R_m, R_i)}{E u'(c_{t+1})}$$

Take the ratio for assets $i$ and $m$:

$$\beta_i = \frac{E R_i - R}{E R_m - R} = \frac{\text{Cov}(R_m, R_i)}{\text{Var}(R_m)}$$

(20)

Note: $\beta_i$ is the coefficient of regressing $R_i$ on $R_m$ using OLS.

This is the famous **CAPM** asset pricing equation.
The risk premium for asset $i$ depends on:

- it’s **beta** (essentially the correlation with the market)
- the market price of risk: $E R_m - R$.

A stock’s beta can be estimated from data on past returns of the stock ($R_i$) and the market (using a broad stock index).

Betas are used to

- Measure the risk of an asset.
- Calculate the required rate of return for investment projects.
- Evaluation of mutual fund managers.
Securities market line

CAPM prediction:

\[
\mathbb{E} R_i = (1 - \beta_i) R + \beta_i \mathbb{E} R_m \\
= R + \beta_i \mathbb{E} \{R_m - R\} 
\]  

(21)  

(22)

If we plot expected returns against \( \beta \)s, we should get a straight line. This is called the securities market line (SML)
The Capital Asset Pricing Model is an elegant theory with profound implications for asset pricing and investor behavior. But how useful is the model given the idealized world that underlies its derivation? There are several ways to answer this question. First, we can examine whether real-world asset prices and investor portfolios conform to the predictions of the model, if not always in a strict quantitative sense, and least in a strong qualitative sense. Second, even if the model does not describe our current world particularly well, it might predict future investor behavior—e.g., as a consequence of capital market frictions being lessened through financial innovation, improved regulation, and increasing capital market integration. Third, the CAPM can serve as a benchmark for understanding the capital market phenomena that cause asset prices and investor behavior to deviate from the prescriptions of the model.

Suboptimal Diversification

Consider the CAPM prediction that investors all will hold the same (market) portfolio of risky assets. One does not have to look far to realize that investors do not hold identical portfolios, which is not a surprise since taxes alone will cause idiosyncratic investor behavior. For example, optimal management of capital gains taxes involves early realization of losses and deferral of capital gains, and so taxable investors might react very differently to changes in asset values depending on when they purchased the asset (Constantinides, 1983). Nevertheless, it will still be a positive sign for the model if most investors hold broadly diversified portfolios. But even here the evidence is mixed. On one hand, popular index funds make it possible for investors to obtain diversification at low cost. On the other hand, many workers hold concentrated ownership of company stock in employee retirement savings plans and many executives hold concentrated ownership of company stock options.

Securities market line: Evidence

Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003

Source: Fama (2004)
Implications

Stocks with higher $\beta$s have higher expected returns, but the relationship is flatter than predicted.
Again: we don’t understand how investors value / measure risk.

▶ a fundamental problem.

Oddly, $\beta$ remains popular, even though it does not work in the data.
Solving for the asset price

We show that the asset price equals the present discounted value of dividends

\[ p_t = \mathbb{E}_t \sum_{j=1}^{\infty} d_{t+j} \text{MRS}(t, t+j) \]  \hspace{1cm} (23)

The discount factor is the MRS, called the \textit{stochastic discount factor}. 

Solving for the asset price

Start from the Euler equation:

\[ u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \]  \hspace{1cm} (24)

Solve for the price:

\[ p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \]  \hspace{1cm} (25)

Replace \( p_{t+1} \) with (25) shifted to \( t + 1 \):

\[
\begin{align*}
p_t &= E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\
& + E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[ \frac{\beta u'(c_{t+2})}{u'(c_{t+1})} \right] (p_{t+2} + d_{t+2}) \right\}
\end{align*}
\]  \hspace{1cm} (26)
Solving for the asset price

The law of iterated expectations:

\[ E_t \{ E_{t+1}(x) \} = E_t(x) \]  
\[ (27) \]

Eliminate the \( E_{t+1} \):

\[ p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + E_t \left\{ \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} (p_{t+2} + d_{t+2}) \right\} \]
\[ (28) \]

Iterate forward for \( T \) periods:

\[ p_t = E_t \left\{ \sum_{j=1}^{T} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} + E_t \left\{ \frac{\beta^{T+1} u'(c_{t+T+1})}{u'(c_{t+T})} (p_{t+T+1} + d_{t+T+1}) \right\} \]
\[ (29) \]
Solving for the asset price

Impose that the last term vanishes in the limit:

\[ p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \]  

(31)

▶ There is no good reason for this assumption!
▶ We will see later: other prices solve the asset pricing equation (bubbles)

The asset price equals the **discounted present value of dividends**.

The stochastic **discount factor** is the marginal rate of substitution.
Example: Log Utility

In the Lucas model, assume: \( u(c) = \ln(c) \). \( K = 1 \).

In equilibrium: \( c_t = d_t \).

\[
MRS_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{\beta d_t}{d_{t+1}}.
\]

The asset pricing equation becomes

\[
p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j d_t}{d_{t+j}} d_{t+j} \right\}
\]

\[
= d_t \frac{\beta}{1 - \beta}
\]
Example: Periodic dividends

In the Lucas model, assume:

- Utility is \( u(c) = c^{1-\sigma} / (1 - \sigma) \).
- \( d_t \) alternates between \( d^H \) and \( d^L \).

Asset pricing equation:

\[
\begin{align*}
p_t & \quad = \sum \beta^j (d_t / d_{t+j})^\sigma d_{t+j} \\
& \quad = d_t^\sigma \sum \beta^j d_{t+j}^{1-\sigma} \\
\end{align*}
\]

On good days, \( p_t \) is pulled up by low \( u'(c') \), but is pushed down by low \( d_{t+1} \).
The Excess Volatility Puzzle

Consider a stock with dividend process $d_t$. Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\}$$

(33)

In the data:

- Dividends are very smooth (a goal of company policy).
- Stock prices are much more volatile than dividends.

But in the theory: stock prices should be the average of future dividends and thus smoother than dividends. This is the flip-side of the Equity Premium Puzzle. See Shiller (1981)
Excess Volatility

Real price (log scale)

- Real S&P 500 index
- Present value of subsequent dividends

Source: FRSBSF Economic Letter Nov 2007
Recall how the asset pricing formula is derived:

We iterate forward on the asset pricing Euler equation

\[ p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \] (34)

We assume that the \( p_{t+1} \) term vanishes in the limit.

What if it does not vanish?

Then any (current) asset price can satisfy the asset pricing equation.

The deviation between \( p_t \) and the fundamental price from (34) is called a bubble.

It is purely a self-fulfilling expectation.
Bubbles: Example

- Consider an asset that pays no dividends.
- Its **fundamental price** is 0.
- Assume that the MRS is constant at \( \frac{\beta u'(c_{t+1})}{u'(c_1)} = 1 \).
- The asset pricing equation is

\[
p_t = E_t p_{t+1} \tag{35}
\]

- One price process that satisfies this: \( p \) doubles with probability \( 1/2 \) and drops to 0 otherwise.
- This satisfies (35) for **any** \( p_t \).
- Bubbles are a possible explanation for asset price volatility.
- Note that the bubble does not offer any excess return opportunities.
Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.


