Example: Overlapping Generations Model

Econ720. Fall 2021. Prof. Lutz Hendricks

1 Government bonds in an OLG model

Demographics: At each date $N_t = (1+n)^t$ households are born.

Preferences are given by

$$(1-\beta)\ln(c_t^y) + \beta\ln(c_{t+1}^o)$$

Endowments: The initial old are endowed with s_0 units of capital. Each young is endowed with one unit of work time.

Technology:

$$C + K' - (1-\delta)K = F(K,L) = K^\alpha L^{1-\alpha}$$

Government: The government only rolls over debt from one period to the next:

$$B_{t+1} = R_t B_t$$

Markets: for goods, bonds, labor, capital rental.

Questions: (a) Solve the household problem for a saving function.

(b) Derive the FOCs for the firm.

(c) Define a competitive equilibrium. Make sure the number of variables equals the number of independent equations.

(d) Derive the law of motion for the capital stock

$$(b_{t+1} + k_{t+1})(1+n) = \beta(1-\alpha)k_t^\alpha \tag{1}$$

, where $b = B/L$.

(e) Derive the steady state capital stock for $b = 0$. Why does it not depend on δ?

(f) Derive the steady state capital stock for $b > 0$.

(g) Can you show that the capital stock is lower in the steady state with positive debt (crowding out)?

Answer: Government bonds

(a) The household solves $\max (1-\beta)\ln(w-s)+\beta\ln(R's)$.
The FOC is $c'/c = R'\beta/(1-\beta)$. Therefore $s=(w-s)\beta/(1-\beta)$ and thus $s=\beta w$.
(b) Firms: This is standard:

$$
\begin{aligned}
r &= f'(k) = \alpha k^{\alpha-1} \\
w &= f(k) - f'(k)k = (1-\alpha)k^{\alpha}
\end{aligned}
$$

where $k = K/L$.
(c) A CE is a list of sequences $(c_t^y, c_t^o, s_t, K_t, L_t, b_t, w_t, r_t)$ that satisfy

- the saving function and the 2 household budget constraints
- the 2 firm FOCs
- goods market clearing: $N_t c_t^y + N_{t-1}c_t^o + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$.
- labor market clearing: $L_t = N_t$
- government budget constraint
- capital markets
  - Define $s_t = b_{t+1}^h + k_{t+1}^h$.
  - Then bond market clearing is $N_t b_{t+t}^h = L_{t+1}b_{t+1}$ or $b_{t+1}^h = (1+n)\, b_{t+1}$.
  - Similarly, for capital we have $N_t k_{t+1}^h = L_{t+1}k_{t+1}$.

(d) Law of motion: This follows directly from the capital market clearing condition together with the equilibrium levels of w and the saving function.
(e) Steady state with $b=0$: From the law of motion:

$$\tilde{k}^{1-\alpha} = \beta(1-\alpha)/(1+n)$$

It does not depend on δ because of log utility: households save a constant fraction of earnings.
(f) Steady state with $b>0$. Now we need to satisfy the law of motion for b: $b'(1+n) = Rb$. In steady state: $R = 1+n$. The steady state capital stock therefore satisfies $\alpha k^{\alpha-1} - \delta = n$ or

$$\hat{k}^{1-\alpha} = \alpha/(n+\delta)$$

Note that the steady state satisfies the Golden Rule. There is some concern that this steady state may not be stable. Imagine that $R > 1+n$. Then b rises ($b' > b$). This may reduce the capital stocks and drive up R even further, etc.
(g) Crowding out: We still have (1), which implies

$$b/k^{\alpha} + k = \frac{\beta\,(1-\alpha)}{1+n} \tag{2}$$

Then $b>0$ implies that $k<\tilde{k}$. Or, the long way round: From the law of motion derived in (d) we have

$$b = \frac{\beta(1-\alpha)}{1+n}k^{\alpha} - k = k\left[\frac{\beta(1-\alpha)}{1+n}k^{\alpha-1} - 1\right]$$

Therefore $b>0$ requires

$$\beta(1-\alpha)/(1+n) > \alpha/(n+\delta)$$

which is what was to be shown.