Overlapping Generations Model

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Introduction

Two approaches for modeling the household sector

1. households live forever (**infinite horizon**)
   tractable
2. households live for finite number of periods (**overlapping generations**)
   can talk about questions where demographics matter

Why choose OLG?

1. Demographic structure matters:
    1.1 Social security and tax analysis (pioneered by Auerbach and Kotlikoff 1987)
    1.2 Human capital: schooling followed by on-the-job learning (e.g., many papers by Heckman and his students)
    1.3 Income or wealth inequality (e.g., Huggett 1996; Huggett et al. 2011)

   These are usually computational many-period models.
2. Analytical tractability:
   Usually two period OLG models.
   With log utility consumption becomes independent of r_{t+1}.
   Easy dynamics because agents behave as if not foward looking.
   E.g., Aghion et al. (2002), Galor (2005), Krueger and Ludwig (2007)

What we do in this section

How to set up and solve an OLG model

Show that the world is **not efficient**: households may save too much.

"Social security" can prevent overaccumulation

We can make households "infinitely lived" by adding altruistic **bequests**.

What we don't do in this section

- We sidestep some technical issues:
  - why is there a representative household?
  - why is there a representative firm?
- We come back to those later.

An OLG Model Without Firms

Steps

We go through the standard steps:

1. Describe the economy: demographics, endowments, preferences, technologies, markets
2. Solve each agent's problem
3. Market clearing
4. Competitive equilibrium

We discuss why we make various modeling choices.

Digression: What makes a good model?

Should a model be "realistic" or "unrealistic?"

Should it be simple or complex?

Digression: Why do we use models?

Or: What do models actually do?

Demographics

Time is discrete and goes on forever.

At each date t a cohort of size

$$N_t = N_0(1+n)^t$$

is born.

Each person lives for two periods.

At each date there are N_t young and N_{t-1} old households.

- $N_t/N_{t-1} = 1+n$

Endowments, Preferences

Endowments

- Young households receive endowments w_t.

Preferences: $u(c_t^y) + \beta u(c_{t+1}^o)$.

- u is strictly concave
- $\beta > 0$ is the discount factor.

Note: Putting anything other than consumption and leisure in preferences is frowned upon (in macro, but not in micro). Why?

Technology

Endowments can be stored.

Storing s_t today yields $f(s_t)$ tomorrow.

- f is strictly concave and increasing

Resource constraint:

- Output = consumption + investment
- $Y = C + I$
- Closed economy: saving = investment; $S = I$

$$\underbrace{N_t w_t + N_{t-1} f(s_{t-1})}_{Y_t} = \underbrace{N_t c_t^y + N_{t-1} c_t^o}_{C_t} + \underbrace{N_t s_t}_{I_t} \tag{1}$$

Technology

Resource constraints

Technological constraints that describe the set of feasible choices.
Contain only quantities (no prices).
Often identical to market clearing conditions.

Markets

Goods are traded in competitive spot markets.

- the price of the good is normalized to 1 for all t
  (why can we do this?)

Households can issue one period bonds with interest rate r_{t+1}.

We are done with the description of the environment.

Next step: solve the household problem.

A Missing Market

Even though there is a bond market, **intergenerational** borrowing and lending is not possible.

The reason: the young at t cannot borrow from the old because the old won't be around at $t+1$ to have their loans repaid.

- If households live for more periods, the problem becomes weaker, but does not go away.

An asset that stays around forever solves this problem

- e.g., money, land, shares

Household Problem

The budget constraints are

$$\begin{aligned} w_t &= c_t^y + s_{t+1} + b_{t+1} \\ c_{t+1}^o &= f(s_{t+1}) + b_{t+1}(1 + r_{t+1}) \end{aligned}$$

Why are there no prices for bonds and s?

Lifetime budget constraint

There really is only one constraint that the household cares about:

At what rate can one exchange c_{t+1}^{o} against c_{t}^{y}?

Substitute out bonds:

$$c_{t+1}^{o} = f(s_{t+1}) + (1+r_{t+1})\left(w_t - c_t^y - s_{t+1}\right) \tag{2}$$

The relative price of old vs. young consumption is the interest rate.

Lifetime budget constraint:

$$w_t + \frac{f(s_{t+1})}{1+r_{t+1}} - s_{t+1} = c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}}$$

Present value of income = present value of spending.

Lagrangian

$$
\begin{aligned}
\Gamma &= u(c_t^y) + \beta u(c_{t+1}^o) \\
&\quad + \lambda_t \{ [w_t - c_t^y - s_{t+1}] - [c_{t+1}^o - f(s_{t+1})]/[1 + r_{t+1}] \}
\end{aligned}
$$

FOCs:

$$
\begin{aligned}
u'(c_t^y) &= \lambda_t \\
\beta u'(c_{t+1}^o) &= \lambda_t/(1 + r_{t+1}) \\
f'(s_{t+1}) &= 1 + r_{t+1}
\end{aligned}
$$

In words...

Interpretation

What does this say in words:

$$f'(s_{t+1}) = 1 + r_{t+1} \tag{3}$$

General insight: the household does things in two steps

1. Maximize lifetime income.
2. Optimally distribute consumption over time.

Euler equation

$$u'(c_t^y) = \beta\,(1 + r_{t+1})\,u'(c_{t+1}^o)$$

Interpretation:

Give up 1 unit of consumption when young and buy a bond.

Marginal cost: $u'\left(c_t^y\right)$

Marginal benefit:

$(1 + r_{t+1})$ units of consumption when old

valued at $\beta u'\left(c_{t+1}^o\right)$

A general point

The household decides sequentially:

1. Make choices to maximize lifetime income (here: choose s and b)
2. Use the Euler equation to decide how to allocate that income over time
   so that marginal utility is "equalized" across periods (adjusting for the incentives to postpone consumption to earn interest)

Household Solution

A vector $(c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})$ which satisfies

- 2 FOCs (an EE and the foc for s)
- 2 budget constraints.

Equilibrium

A CE is an allocation

and a price system

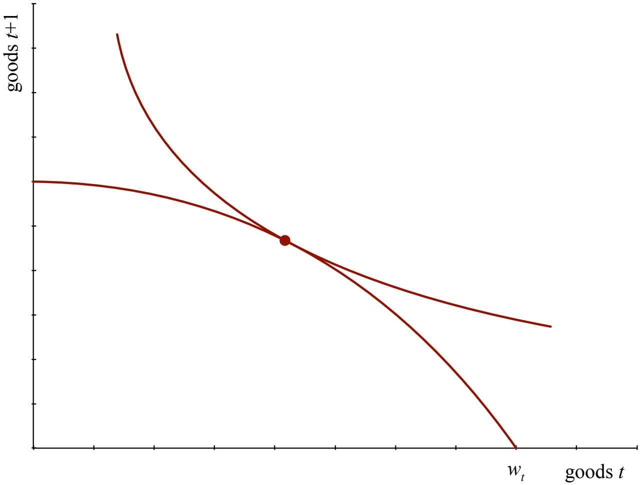
that satisfy:

We are done with the definition of equilibrium.

Next step: characterize equilibrium.

Characterization

There is no trade in equilibrium $(b_t = 0)$



A Production Economy

A Production Economy

The model is modified by adding firms who rent capital and labor from households.

The endowment w is now interpreted as labor earnings.

Households supply one unit of labor inelastically to firms when young.

Capital depreciates at rate δ.

This is a standard setup in many macro models.

Model Elements

- Unchanged: demographics, preferences
- Endowments:
  - at $t = 0$ each old household owns k_0 units of capital
  - each young has 1 unit of work time
- Technology

$$\underbrace{F(K_t, L_t)}_{Y_t} = C_t + \underbrace{K_{t+1}(1-\delta)K_t}_{I_t} \tag{4}$$

  - constant returns to scale
  - Inada conditions
- Markets:
  - goods (numeraire), capital rental (q), labor rental (w)

Capital and timing

At the start of period t:

- the economy is endowed with the capital stock K_t

During t:

- labor L_t and K_t are used to produce Y_t
- households eat C_t and save $S_t = Y_t - C_t$
- fraction δ of the capital stock disappears

At the end of t:

- $K_{t+1} = (1-\delta)K_t + S_t$ is taken into $t+1$

Notes

Representative household

- All households are the same.
- So we talk as if there were only 1 household, who behaves competitively.

The household owns everything

- The firm rents capital from the household in each period
- That makes the firms' problem static (easy)
- It is usually convenient to pack all dynamic decisions into 1 agent

In this model, who owns the capital makes no difference - why not?

Welfare theorems

Roughly speaking:

> *If all markets are competitive and there are no externalities or distortionary taxes*
>
> *then*
>
> *Any competitive equilibrium is Pareto optimal.*

Though there is a technical wrinkle that derails efficiency in this model...

Households

Budget constraints:

$$
\begin{aligned}
w_t &= c_t^y + s_{t+1} + b_{t+1} \\
c_{t+1}^o &= e^o + (s_{t+1} + b_{t+1})(1 + r_{t+1})
\end{aligned}
$$

e^o: any other income received when old (currently 0)

There are no profits b/c the technology has constant returns to scale.

Lifetime budget constraint

Combine the 2 budget constraints:

$$w_t - c_t^y = \left(c_{t+1}^o - e^o\right) / [1 + r_{t+1}]$$

or

$$W_t = \underbrace{w_t + \frac{e^o}{1 + r_{t+1}}}_{\text{p.v. of income}} = \underbrace{c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}}}_{\text{p.v. of consumption}} \tag{5}$$

W_t: present value of lifetime earnings

Permanent Income Hypothesis

The lifetime budget constraint only depends on W_t, not on timing of income over life.

$$c^o_{t+1} = (1 + r_{t+1}) \left(W_t - c^y_t \right) \tag{6}$$

Therefore, the optimal consumption path only depends on W_t.

This is a somewhat general implication that has been tested many times. example:

- One example: Hsieh (2003) [Nice example of using a natural experiment to test a theory.]

Overall, the evidence seems favorable.

Lagrangian

$$\begin{aligned}\Gamma &= u(c_t^y) + \beta u(c_{t+1}^o) \\ &\quad + \lambda\{W_t - c_t^y - c_{t+1}^o/[1 + r_{t+1}]\}\end{aligned}$$

FOCs:

$$\begin{aligned}u'(c_t^y) &= \lambda \\ \beta u'(c_{t+1}^o) &= \lambda/(1 + r_{t+1})\end{aligned}$$

In words...

Households

Euler:

$$u'(c_t^y) = \beta(1 + r_{t+1})u'(c_{t+1}^o)$$

Solution: A vector $(c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})$ that satisfies 2 budget constraints and 1 EE.

We lack one equation! Why?

Consumption theory basics

The Euler equation + present value budget constraint are the essence of the theory of consumption.

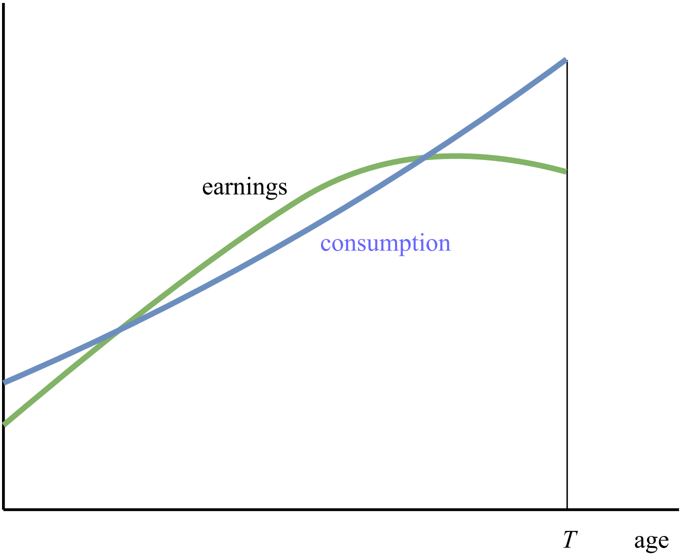
- The Euler equation gives the "slope" of the age-consumption profile.
- The budget constraint gives the level.

E.g., log utility:

- $u'(c) = 1/c$
- $c_{t+1}/c_t = \beta(1 + r_{t+1})$

Intuition: the effect of shocks... (graph)

Consumption and shocks



Testable implications

Strong, testable implications

- all households have the same consumption growth rate
- when income is received over the life-cycle does not matter

The theory seems hopelessly simplistic.

But it gets better when income is stochastic (we study such models later).

Firms

Firms maximize **current period** profits taking factor prices (q, w) as given.

$$\max F(K,L) - wL - qK$$

Note: When firms own something (e.g., capital), they solve an intertemporal problem.

FOCs:

$$\begin{aligned} q &= F_K(K,L) \\ w &= F_L(K,L) \end{aligned}$$

The **solution** to the firm's problem is a pair (K, L) so that the 2 FOCs hold.

Firms

A wrinkle: We assume constant returns to scale.

The size of the firm is indeterminate (why?)

The FOCs only determine K/L (not K and L separately).

Firms: Intensive form

It is convenient to write the production function in **intensive form**:

$$\begin{aligned} F(K,L) &= LF(K/L,1) \\ &= Lf(k^F) \end{aligned}$$

where $k^F = K/L$ and

$$f(k^F) = F(k^F,1)$$

Firms: Intensive form

Now the factor prices are

$$F_K = \frac{\partial Lf\left(K/L\right)}{\partial K} = Lf'(k^F)(1/L)$$

and

$$\begin{aligned} F_L &= \tfrac{\partial Lf(K/L)}{\partial L} &&= f(k^F) + Lf'(k^F)(-K/L^2) \\ &= && f(k^F) - f'(k^F)k^F \end{aligned}$$

Therefore:

$$\begin{aligned} q &= f'(k^F) \\ w &= f(k^F) - k^F f'(k^F) \end{aligned}$$

Important: q is the rental price of capital, which differs from the interest rate r.

Market clearing

Capital rental: $N_t s_{t+1} = K_{t+1}$

Labor rental: $L_t = N_t$

Bonds: $b_t = 0$

Goods: resource constraint

Note: The saving of the young is the entire capital stock next period.

- Undepreciated capital goes to the old who do not save.

Competitive Equilibrium

An allocation: $(c_t^y, c_t^o, s_t, b_t, K_t, L_t)$

Prices: (q_t, r_t, w_t)

That satisfy:

1. Household: 3
2. Firm: 2
3. Market clearing: 4

We have 9 objects and 9 equations – one is missing.

We need an accounting identity linking r and q:

- The household receives $1 + r_{t+1} = q_{t+1} + 1 - \delta$ per unit of capital.
- Therefore, $r = q - \delta$.

Competitive equilibrium

We could also write everything in terms of $k^F = K/L$ and drop L from the CE definition.

Then

- Capital market clearing:

$$s_{t+1} = \frac{K_{t+1}}{N_t} = k_{t+1}(1+n) \tag{7}$$

- Goods market clearing:

$$f\left(k_t^F\right) + (1-\delta)k_t^F = c_t^y + c_t^o/(1+n) + k_{t+1}(1+n) \tag{8}$$

Reading

- Acemoglu (2009), ch. 9.
- Krueger, "Macroeconomic Theory," ch. 8
- Ljungqvist and Sargent (2004), ch. 9 (without the monetary parts).
- McCandless and Wallace (1991) and De La Croix and Michel (2002) are book-length treatments of overlapping generations models.

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