Problem Set 2: OLG Models with Money

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1 Money and Heterogeneity

Consider a two-period OLG model with fiat money.

Demographics: In each period $N_t = (1+n)^t$ persons are born. Each lives for 2 periods. Half of the agents are of type I, the other half of type II.

Endowments: The initial old hold M_0 units of money, evenly distributed across agents. Each person is endowed with (e_i^y, e_i^o) units of consumption when (young, old).

Preferences: $\ln(c_t^y) + \beta \ln(c_{t+1}^o)$.

Technology: Goods can only be eaten the day they drop from the sky.

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_t = M_{t-1} + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu) M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

Questions:

1. Define a competitive equilibrium.
2. Derive the household consumption function.
3. Derive a difference equation for the equilibrium interest rate when $\mu = 0$.
4. Is the monetary steady state dynamically efficient?

1.1 Answer: Money and Heterogeneity

It is important to be clear about the timing. At the start of period t, the money stock is M_{t-1}. Then the old receive the transfer x_t, which raises the money stock to M_t. Let $m_{i,t}^d$ be the real, per capita money holdings of the young in t (and thus of the old at the start of $t+1$).

1. Equilibrium:

   Objects: $\left\{c_{i,t}^y, c_{i,t}^o, m_{i,t}^d, x_t, p_t, M_t\right\}$

   Equations:

   - household: Euler equation and 2 budget constraints. Young: $e_i^y = c_{i,t}^y + m_{i,t}^d$. Old: $e_i^o + m_{i,t}^d p_t / p_{t+1} + x_{t+1}$.
   - government: 2 laws of motion for M_t
   - market clearing:
     - goods: $\sum_i e_i^y + e_i^o / (1+n) = \sum_i c_{i,t}^y + c_{i,t}^o / (1+n)$

– money: $M_t = p_t \sum_i N_{i,t} m_{i,t}^d$ (here $N_{i,t} = N_t/2$).

2. Consumption function: Euler + lifetime budget constraint implies that

$$c_{it}^y (1+\beta) = W_{it} = e_i^y + (e_i^o + x_{t+1}) / R_{t+1} \tag{1}$$

and

$$c_{it}^o = \frac{\beta}{1+\beta} W_{it} R_{t+1} \tag{2}$$

where $R_{t+1} = 1/(1+\pi_{t+1})$ and $1+\pi_{t+1} = p_{t+1}/p_t$.

3. Implicit solution for the equilibrium interest rate when $x = 0$: Start from goods market clearing

$$\sum_i W_{it}/(1+\beta) + W_{i,t-1} \frac{\beta R_t}{(1+\beta)(1+n)} = E = \sum_i e_i^y + e_i^o/(1+n) \tag{3}$$

Now sub in the definitions for $W_{i,t}$:

$$\sum_i \frac{e_i^y + e_i^o/R_{t+1}}{1+\beta} + \sum_i \frac{e_i^y + e_i^o/R_t}{1+n} \frac{\beta R_t}{1+\beta} = E \tag{4}$$

This produces a difference equation in R_t.

1. Steady state with money:

   We need $m = M/(NP)$ to be constant over time. Therefore, $1+\pi = (1+\mu)/(1+n)$. With $\mu > 0$, the monetary economy must be dynamically *in*efficient. With $\mu = 0$, we get the Golden Rule and the economy is efficient. The same could be derived by setting R constant in the difference equation above.

2 Money in the Utility Function in an OLG Model

Demographics: In each period a cohort of constant size N is born. Each person lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M. No new money is ever issued. The young are endowed with one unit of work time.

Preferences: $u(c_t^y) + \beta\, u(c_{t+1}^o) + v(m_t^d/p_t)$. Assume $v' > 0$. Agents derive utility from real money balances as defined below.

Technology: Output is produced with a constant returns to scale production function $F(K_t, L_t)$. The resource constraint is standard. Capital depreciates at rate δ.

Markets: There are spot markets for goods (price p_t), money, labor (wage w_t), and capital rental (price q_t).

Timing:

- The old enter period t holding money M and capital K_t.
- Production takes place.
- The old sell money to the young. m_t^d is the nominal per capita money holding of a young person.
- Consumption takes place.

Questions:

1. Derive a set of 4 equations that characterize optimal household behavior. Show that the household's first-order conditions imply rate of return dominance, i.e., the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium).

2. Solve the firm's problem.

3. Define a competitive equilibrium.

4. Assume that the utility functions u and v are logarithmic. Solve *in closed form* for the household's money demand function, $m_t^d/p_t = \varphi(w_t, r_{t+1}, \pi_{t+1})$, and for its saving function, $s_{t+1} = \phi(w_t, r_{t+1}, \pi_{t+1})$. $\pi_{t+1} \equiv p_{t+1}/p_t$.

2.1 Answer: Money in the Utility Function in an OLG Model

1. The household solves max $u(c_t^y) + \beta\, u(c_{t+1}^o) + v(m_t^d/p_t)$ subject to the budget constraints $w_t = c_t^y + m_t^d/p_t + s_{t+1}$ and $c_{t+1}^o = s_{t+1}\, R_{t+1} + m_t^d/p_{t+1}$. This is most easily set up using a lifetime budget constraint:

$$w_t - c_t^y - m_t^d/p_t = s_{t+1} = \frac{c_{t+1}^o - m_t^d/p_{t+1}}{R_{t+1}} \tag{5}$$

   The first-order conditions are:

$$\begin{aligned} u'(c_t^y) &= \beta\, R_{t+1}\, u'(c_{t+1}^o) \\ v'(m_t^d/p_t) &= \beta\, u'(c_{t+1}^o)\, [R_{t+1} - p_t/p_{t+1}] \end{aligned}$$

   Interpretation: If the household saves one unit as money rather than capital, he gains v', but loses some old consumption because the rate of return of money is lower than that of capital. Note that this implies rate of return dominance because positive marginal utility requires $R_{t+1} > p_t/p_{t+1}$. A solution to the household problem is a vector $(c_t^y, c_{t+1}^o, m_t^d, s_{t+1})$ which solves the 2 first-order conditions and the 2 budget constraints.

2. The firm's problem is standard with first order conditions $r_t = F_K(K_t, L_t)$ and $w_t = F_L(K_t, L_t)$.

3. A competitive equilibrium is an allocation $(c_t^y,\, c_{t+1}^o, m_t^d,\, s_{t+1},\, K_t, L_t)$ and a price system $(R_t,\, r_t,\, w_t,\, p_t)$ that satisfies 4 household FOCs, 2 firm FOCs, the identity $R_t = 1 + r_t - \delta$, and market clearing. The capital market clears if $N s_{t+1} = K_{t+1}$. The money market clears if $m_t^d = m_0$. Goods market clearing requires that $F(K_t, L_t) + (1-\delta)\, K_t = N c_t^y + N c_t^o + K_{t+1}$. We have 10 variables and 11 equations, one of which is redundant by Walras' law.

4. (d) With log utility the household first-order conditions become $c_{t+1}^o = \beta\, R_{t+1}\, c_t^y = \beta\, m_t/p_t\, (R_{t+1} - p_t/p_{t+1})$. Define the inflation rate $\pi_{t+1} = p_{t+1}/p_t$. Then the budget constraint 5 together with the first-order condition for c_t^y imply

$$\frac{c_{t+1}^o}{R_{t+1}}\,(1 + 1/\beta) = w_t - \frac{m_t}{p_t}\,(1 - 1/\pi_{t+1} R_{t+1})$$

Substituting out c^o_{t+1} and simplifying yields the *money demand function*

$$w_t = \frac{m_t}{p_t} (2 + \beta) (1 - 1/\pi_{t+1} R_{t+1}).$$

This has sensible properties. A higher nominal interest rate reduces money demand. If the household is more patient or richer, more money is held. To solve for the saving function:

$$\begin{aligned}
s_{t+1} &= \frac{c^o_{t+1} - m_t/(p_t\,\pi_{t+1})}{R_{t+1}} \\
&= \frac{\beta\, m_t/p_t\,(R_{t+1} - 1/\pi_{t+1}) - m_t/(p_t\,\pi_{t+1})}{R_{t+1}} \\
&= \frac{m_t}{p_t}\,(\beta\,(1 - 1/\pi_{t+1} R_{t+1}) - 1/\pi_{t+1} R_{t+1})
\end{aligned}$$

Substitute out m_t/p_t using the money demand function.

$$s_{t+1} = w_t \frac{\beta\,(\pi_{t+1} R_{t+1} - 1) - 1}{(2 + \beta)\,(\pi_{t+1} R_{t+1} - 1)}$$

From the saving function one could derive a single equation that characterizes the steady state capital stock. In steady state, $\pi = 1$; otherwise real money balances would not be constant. Apply this to the saving function and equate $s = K$. Then note that w and R are functions of K.