Problem Set 2: OLG Models with Money  
Econ720. Fall 2019. Prof. Lutz Hendricks. October 15, 2019

1 Money and Heterogeneity

Consider a two-period OLG model with fiat money.

Demographics: In each period \( N_t = (1 + n)^t \) persons are born. Each lives for 2 periods. Half of the agents are of type I, the other half of type II.

Endowments: The initial old hold \( M_0 \) units of money, evenly distributed across agents. Each person is endowed with \( (e^y_i, e^o_i) \) units of consumption when (young, old).

Preferences: \( \ln(c^y_t) + \beta \ln(c^o_{t+1}) \).

Technology: Goods can only be eaten the day they drop from the sky.

Government: The government pays a lump-sum transfer of \( x_t^p \) units of money to each old person: \( M_t = M_{t-1} + N_{t-1}x_t^p \). The aggregate money supply grows at the constant rate \( \mu \): \( M_{t+1} = (1 + \mu)M_t \).

Markets: In each period, agents buy/sell goods and money in spot markets.

Questions:

1. Define a competitive equilibrium.
2. Derive the household consumption function.
3. Derive a difference equation for the equilibrium interest rate when \( \mu = 0 \).
4. Is the monetary steady state dynamically efficient?

1.1 Answer: Money and Heterogeneity

It is important to be clear about the timing. At the start of period \( t \), the money stock is \( M_{t-1} \). Then the old receive the transfer \( x_t \), which raises the money stock to \( M_t \). Let \( m^d_{i,t} \) be the real, per capita money holdings of the young in \( t \) (and thus of the old at the start of \( t + 1 \)).

1. Equilibrium:

   Objects: \( \{c^y_{i,t}, c^o_{i,t}, m^d_{i,t}, x_t, p_t, M_t\} \)

   Equations:
   - household: Euler equation and 2 budget constraints. Young: \( e^y_i = c^y_{i,t} + m^d_{i,t} \). Old: \( e^o_i + m^d_{i,t}p_t/p_{t+1} + x_{t+1} \).
   - government: 2 laws of motion for \( M_t \)
   - market clearing:
     - goods: \( \sum_i e^y_i + e^o_i / (1 + n) = \sum_i c^y_{i,t} + c^o_{i,t} / (1 + n) \)
1. Steady state with money:

We need \( m = M/(NP) \) to be constant over time. Therefore, \( 1 + \pi = (1 + \mu) / (1 + n) \). With \( \mu > 0 \), the monetary economy must be dynamically inefficient. With \( \mu = 0 \), we get the Golden Rule and the economy is efficient. The same could be derived by setting \( R \) constant in the difference equation above.

2. Consumption function: Euler + lifetime budget constraint implies that

\[
c_y^t (1 + \beta) = W_t = (e_y^o + e_x^o + s_{t+1}) / R_{t+1}
\]

and

\[
e^o_i = \frac{\beta}{1 + \beta} W_t R_{t+1}
\]

where \( R_{t+1} = 1 / (1 + \pi_{t+1}) \) and \( 1 + \pi_{t+1} = p_{t+1} / p_t \).

3. Implicit solution for the equilibrium interest rate when \( x = 0 \): Start from goods market clearing

\[
\sum_i W_{it} / (1 + \beta) + W_{it-1} \frac{\beta R_t}{(1 + \beta) (1 + n)} = E = \sum_i e^y_i + e^o_i / (1 + n)
\]

Now sub in the definitions for \( W_{it} \):

\[
\sum_i e^y_i + e^o_i / R_{t+1} + \sum_i e^y_i + e^o_i / R_t \beta R_t / (1 + \beta) = E
\]

This produces a difference equation in \( R_t \).

2 Money in the Utility Function in an OLG Model

Demographics: In each period a cohort of constant size \( N \) is born. Each person lives for 2 periods.

Endowments: The initial old hold capital \( K_0 \) and money \( M \). No new money is ever issued. The young are endowed with one unit of work time.

Preferences: \( u(c^o_t) + \beta u(c^o_{t+1}) + v(m^d_t / p_t) \). Assume \( v' > 0 \). Agents derive utility from real money balances as defined below.

Technology: Output is produced with a constant returns to scale production function \( F(K_t, L_t) \).

The resource constraint is standard. Capital depreciates at rate \( \delta \).

Markets: There are spot markets for goods (price \( p_t \)), money, labor (wage \( w_t \)), and capital rental (price \( q_t \)).

Timing:

- The old enter period \( t \) holding money \( M \) and capital \( K_t \).
- Production takes place.
- The old sell money to the young. \( m^d_t \) is the nominal per capita money holding of a young person.
- Consumption takes place.
Questions:

1. Derive a set of 4 equations that characterize optimal household behavior. Show that the household’s first-order conditions imply rate of return dominance, i.e., the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium).

2. Solve the firm’s problem.

3. Define a competitive equilibrium.

4. Assume that the utility functions \( u \) and \( v \) are logarithmic. Solve in closed form for the household’s money demand function, \( m_t^d/p_t \), and for its saving function, \( s_{t+1} = \phi(w_t, r_{t+1}, \pi_{t+1}) \), and for its saving function, \( \pi_{t+1} \equiv p_{t+1}/p_t \).

2.1 Answer: Money in the Utility Function in an OLG Model

1. The household solves
\[
\text{max } \sum_{t=0}^{\infty} \beta^t w_t \left( c_t^y - m_t^d/p_t \right)^{\gamma} - \phi(w_t, r_{t+1}, \pi_{t+1})
\]
subject to the budget constraints
\[ w_t = c_t^y + m_t^d/p_t + s_{t+1} \]
and the first-order conditions are:
\[
\frac{u'(c_t^y)}{u'(c_t^o)} = \beta \frac{R_{t+1} u'(c_{t+1})}{u'(c_{t+1})} \quad \text{and} \quad \frac{v'(m_t^d/p_t)}{v'(m_{t+1}^d/p_{t+1})} = \beta \frac{R_{t+1} - p_t/p_{t+1}}{R_{t+1}}
\]
Interpretation: If the household saves one unit as money rather than capital, he gains \( v' \), but loses some old consumption because the rate of return of money is lower than that of capital. Note that this implies rate of return dominance because positive marginal utility requires
\[ R_{t+1} > p_t/p_{t+1} \]
A solution to the household problem is a vector \( (c_t^y, c_{t+1}^o, m_t^d, s_{t+1}) \) which solves the 2 first-order conditions and the 2 budget constraints.

2. The firm’s problem is standard with first order conditions \( r_t = F_K(K_t, L_t) \) and \( w_t = F_L(K_t, L_t) \).

3. A competitive equilibrium is an allocation \( (c_t^y, c_{t+1}^o, m_t^d, s_{t+1}, K_t, L_t) \) and a price system \( (R_t, r_t, w_t, p_t) \) that satisfies 4 household FOCs, 2 firm FOCs, the identity \( R_t = 1 + r_t - \delta \), and market clearing. The capital market clears if \( Ns_{t+1} = K_{t+1} \). The money market clears if \( m_t^d = m_0 \). Goods market clearing requires that
\[ F(K_t, L_t) + (1 - \delta) K_t = Nc_t^y + Nc_{t+1}^o + K_{t+1} \]
We have 10 variables and 11 equations, one of which is redundant by Walras’ law.

4. (d) With log utility the household first-order conditions become \( c_{t+1}^o = \beta R_{t+1} c_t^y = \beta m_t/p_t \left( R_{t+1} - p_t/p_{t+1} \right) \). Define the inflation rate \( \pi_{t+1} = p_{t+1}/p_t \). Then the budget constraint 5 together with the first-order condition for \( c_t^y \) imply
\[
\frac{c_{t+1}^o}{R_{t+1}} (1 + 1/\beta) = w_t - \frac{m_t}{p_t} \left( 1 - 1/\pi_{t+1} R_{t+1} \right)
\]
Substituting out $c^t_{t+1}$ and simplifying yields the *money demand function*

$$w_t = \frac{m_t}{p_t} \left( 2 + \beta \right) \left( 1 - \frac{1}{\pi_{t+1} R_{t+1}} \right).$$

This has sensible properties. A higher nominal interest rate reduces money demand. If the household is more patient or richer, more money is held. To solve for the saving function:

$$s_{t+1} = \frac{c^t_{t+1} - m_t/(p_t \pi_{t+1})}{R_{t+1}}$$

$$= \frac{\beta m_t/p_t \left( R_{t+1} - 1/\pi_{t+1} \right) - m_t/(p_t \pi_{t+1})}{R_{t+1}}$$

$$= \frac{m_t}{p_t} \left( \beta \left( 1 - 1/\pi_{t+1} R_{t+1} \right) - 1/\pi_{t+1} R_{t+1} \right)$$

Substitute out $m_t/p_t$ using the money demand function.

$$s_{t+1} = w_t \frac{\beta \left( \pi_{t+1} R_{t+1} - 1 \right) - 1}{(2 + \beta) \left( \pi_{t+1} R_{t+1} - 1 \right)}$$

From the saving function one could derive a single equation that characterizes the steady state capital stock. In steady state, $\pi = 1$; otherwise real money balances would not be constant. Apply this to the saving function and equate $s = K$. Then note that $w$ and $R$ are functions of $K$. 

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