

# The Growth Model in Continuous Time

## Competitive Equilibrium

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# Competitive Equilibrium

- ▶ Firms solve the same problem as in the Solow model.
- ▶ We add a government that imposes lump-sum taxes to finance government spending.
- ▶ The budget constraint is  $\tau_t = G_t$ .

# Households

$$\max \int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (1)$$

subject to:  $k_0$  given, the TVC, and the budget constraint

$$\dot{k}_t = w_t + (r_t - \delta - n)k_t - c_t - \tau_t \quad (2)$$

# Households

Hamiltonian:

$$H = u(c) + \lambda[w + (r - \delta - n)k - c - \tau] \quad (3)$$

First-order conditions

$$\partial H / \partial c = 0 \Rightarrow u'(c) = \lambda \quad (4)$$

$$\begin{aligned} \dot{\lambda} &= (\rho - n)\lambda - \partial H / \partial k \\ &= \lambda[\rho - n - (r - \delta - n)] \\ &= \lambda(\rho - r + \delta) \end{aligned}$$

Transversality:

$$\lim_{t \rightarrow \infty} e^{-(\rho - n)t} \lambda_t k_t = 0 \quad (5)$$

# Households

Eliminate  $\lambda$ :

$$u''(c)\dot{c} = \dot{\lambda} \quad (6)$$

Substitute into the law of motion for  $\lambda$ :

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - r]$$

or

$$g_c = (r - \delta - \rho)/\sigma \quad (7)$$

Solution: Functions  $c_t, k_t$  that solve the Euler equation, the budget constraint, and the boundary conditions.

# Competitive Equilibrium

Objects: Functions  $c_t, k_t, \tau_t, w_t, r_t$ .

Equilibrium conditions:

- ▶ Household (2)
- ▶ Firm (2)
- ▶ Government (1)
- ▶ Market clearing (1)

# Dynamics

Simplify to obtain two differential equations:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \quad (8)$$

$$\dot{k} = f(k) - (n + \delta)k - c - G \quad (9)$$

The planning solution and the CE coincide (with  $G = 0$ ).

## Detrending the Model



## Detrending a model

- ▶ Consider the Cass Koopmans model with productivity growth:

$$\max \int_0^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (10)$$

$$\dot{k}_t = F(k_t, A_t) - (n + \delta)k_t - c_t \quad (11)$$

with

$$A_t = e^{gt} \quad (12)$$

- ▶ What does the Planner's solution look like?
- ▶ The problem: the model has no steady state.
- ▶ How can we analyze its dynamics?

## Approach 1: Solve and detrend

- ▶ Unchanged: the Planner's optimality conditions in terms of original variables:

$$\dot{c}/c = \frac{\frac{\partial F(k,A)}{\partial k} - n - \delta - (\rho - n)}{\sigma(c)} \quad (13)$$

- ▶ But we cannot draw the phase diagram without a steady state.
- ▶ Solution: detrend the variables to make them stationary.
  1. Find the balanced growth rate for each variable.
  2. Divide each variable by a scale factor that grows at its balanced growth rate.

## Balanced growth rates

- ▶ The same as in the Solow model with growth:

$$g(c) = g(k) = g \quad (14)$$

- ▶ Define the detrended variables:

$$\tilde{c}_t = c_t/A_t \quad (15)$$

$$\tilde{k}_t = k_t/A_t \quad (16)$$

- ▶ Law of motion:

$$\begin{aligned} g(\tilde{k}) &= g(k) - g \\ &= \frac{F(\tilde{k}, 1)A - (n + \delta)\tilde{k}A - \tilde{c}A}{k} - g \\ d\tilde{k}/dt &= f(\tilde{k}) - (n + \delta + g)\tilde{k} - \tilde{c} \end{aligned} \quad (17)$$

## Detrended first-order conditions

- ▶ Optimality conditions in terms of detrended variables:

$$\begin{aligned}\frac{d\tilde{c}/dt}{\tilde{c}} &= \frac{\dot{c}}{c} - g \\ &= \frac{f'(\tilde{k}) - \delta - \rho}{\sigma(c)} - g\end{aligned}\tag{18}$$

- ▶ This is true because

$$\frac{\partial F(k, A)}{\partial k} = \frac{\partial F(\tilde{k}A, A)}{\partial \tilde{k}} \frac{\partial \tilde{k}}{\partial k} = Af'(\tilde{k}) \frac{1}{A}\tag{19}$$

## Detrended first-order conditions

- ▶ Assume CRRA preferences:

$$u(c) = c^{1-\sigma} / (1-\sigma) \quad (20)$$

- ▶ Then  $\sigma(c) = \sigma$  is constant.
- ▶ **CRRA is required for balanced growth** - an important result.
  - ▶ Otherwise  $\sigma(c)$  is not constant.

## Model Solution

Functions of time  $c_t, k_t$  that satisfy:

1. Euler equation

$$g(\tilde{c}) = \frac{f'(\tilde{k}) - \delta - \rho}{\sigma(c)} - g \quad (21)$$

2. Resource constraint

$$d\tilde{k}/dt = f(\tilde{k}) - (n + \delta + g)\tilde{k} - \tilde{c} \quad (22)$$

3. Initial condition:  $\tilde{k}_0$  given
4. TVC  $\lim_{t \rightarrow \infty} e^{-(\rho-n)t} u'(c_t) k_t = 0$ .

With  $u'(c_t) = c_t^{-\sigma} = \tilde{c}_t^{-\sigma} e^{-\sigma g t}$  and  $k_t = \tilde{k}_t e^{g t}$ , this becomes

$$\lim_{t \rightarrow \infty} e^{-(\rho-n)t} e^{(1-\sigma)g t} u'(\tilde{c}_t) \tilde{k}_t = 0 \quad (23)$$

## Approach 2: Detrend and solve

- ▶ Steps:
  1. Find balanced growth rates - as before.
  2. Write the economy in detrended variables.
  3. Take the first-order conditions.
  4. Define the solution.
  5. Convert back into (undetrended) variables.
- ▶ This is useful for solution methods that only work on stationary problems (such as DP).
- ▶ Exercise: show that this yields the same answer for the growth model.

# Detrending the Model

## Summary

In the growth model, optimality conditions change only by adding the 2 occurrences of  $g$ :

$$g(\tilde{c}) = \frac{f'(\tilde{k}) - \delta - \rho}{\sigma} - g \quad (24)$$

$$d\tilde{k}/dt = f(\tilde{k}) - (n + \delta + g)\tilde{k} - \tilde{c} \quad (25)$$



# Detrending the Model

Why do we care?

1. The balanced growth  $\tilde{k}$  now depends on preferences:

$$g(\tilde{c}) = 0 \Rightarrow f(\tilde{k}) = \delta + \rho + \sigma g \quad (26)$$

2. We see that preferences must be CRRA for a steady state to exist.
3. Quantitative differences.

# Reading

- ▶ Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- ▶ Barro and Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- ▶ Blanchard and Fischer (1989), ch. 2
- ▶ Romer (2011), ch. 2A
- ▶ Phase diagram: Barro and Martin (1995), ch. 2.6

## References I

- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.
- Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.
- Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.