The Growth Model in Continuous Time (Ramsey Model)

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The Growth Model in Continuous Time

We add optimizing households to the Solow model.

We first study the planner's problem, then the CE.

Planning Problem

Planning Problem

The social planner maximizes

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{1}$$

subject to the resource constraint

$$\begin{aligned} \dot{k}_t &= f(k_t) - (n+\delta)k_t - c_t && (2) \\ & k_0 \text{ given} && (3) \\ k_t &\geq 0 && (4) \end{aligned}$$

Planning Problem

The current value Hamiltonian is

The state is k and the control is c.

The optimality conditions are

Planner: TVC

The TVC is:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)k(t) = 0 \tag{5}$$

To check this:

- we need u and $g(k,c)$ to be monotone
- u is obvious.
- $g(k,c) = f(k) - c - (n+\delta)k$ is monotone in c but not k.
- However, we "know" that k never rises above the golden rule point where $f'(k) = \delta$ - unless $k(0)$ is too high.
- Then g is increasing in k.

Sufficiency

This is an example where the easiest (1st) set of sufficiency conditions applies:

- u is strictly concave in c (only).
- $g(k,c)$ is jointly concave in k and c.

First order conditions are sufficient.

Planner: Solution

A solution consists of functions of time

$$c_t, k_t, \mu_t$$

that satisfy:

1. The first-order conditions (2)
2. The resource constraint
3. The boundary conditions k_0 given and the TVC

$$\lim e^{-(\rho-n)t}\mu_t k_t = 0 \tag{6}$$

Planner: Euler Equation

We eliminate the multiplier, starting from

$$
\begin{aligned}
g(\mu) &= -\left[f'(k) - \delta - \rho\right] &(7)\\
&= g\left(u'(c)\right) &(8)\\
&= -\sigma g(c) &(9)
\end{aligned}
$$

where σ is the elasticity of u' w.r.to c:

$$
\begin{aligned}
\sigma &= -u''_c c/u' &(10)\\
&= -\frac{du'(c)}{dc}\frac{c}{u'(c)} &(11)
\end{aligned}
$$

Therfore:

$$\boxed{g(c) = [f'(k) - \delta - \rho]/\sigma} \quad (12)$$

More direct derivation

Differentiating the FOC yields

$$\dot{\mu} = u''(c)\dot{c} \tag{13}$$

and therefore

$$\begin{aligned} \dot{\mu}/\mu &= u''(c)\dot{c}/u'(c) && (14) \\ &= -\left[f'(k) - \delta - \rho\right] && (15) \end{aligned}$$

Note on σ

$$\sigma = -u''_c c/u' \quad (16)$$

$$= -\frac{du'(c)}{dc}\frac{c}{u'(c)} \quad (17)$$

σ is

1. the elasticity of marginal utility w.r.to c
2. the inverse intertemporal elasticity of substitution
3. the coefficient of relative risk aversion

σ is a key parameter for asset pricing, business cycle volatilities, ...

$u(c) = c^{1-\phi}/1-\phi$ implies $\sigma = \phi$.

Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho]/\sigma \tag{18}$$

Recall the discrete time version:

$$\frac{c_{t+1}}{c_t} = [\beta R]^{1/\sigma} \tag{19}$$

The same idea:

- consumption growth rises with the interest rate
- declines with the discount rate
- σ governs how responsive consumption growth is

Planner: Summary

- The planner's problem solves for functions of time $c(t)$ and $k(t)$.
- These satisfy two differential equations

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \tag{20}$$

$$\dot{k} = f(k) - (n+\delta)k - c \tag{21}$$

and two boundary conditions

$$k_0 \text{ given}$$

$$\lim_{t\to\infty} \beta^t u'(c(t))\; k(t) = 0$$

- How can we analyze the dynamics of this system?

Phase Diagram

Phase Diagram

Phase diagrams can be used to analyze the dynamics of systems of 2 differential equations.

Consider the example

$$\begin{aligned}\dot{x} &= A - ax + by \\ \dot{y} &= B + cx - dy\end{aligned}$$

Assume $a, b, c, d > 0$.

Basic Idea

$$\dot{x} = A - ax + by \tag{22}$$

This divides the (x, y) plane into two regions:

- one where x rises (moving east over time)
- one where x falls (moving west over time)

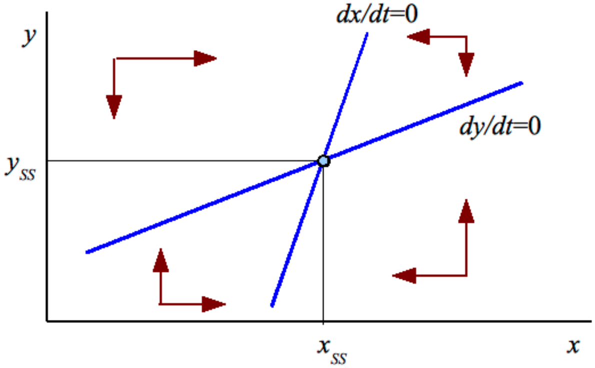
$$(\dot{y} = B + cx - dy) \tag{23}$$

also divides the plane into two regions

- one where y rises (moving north over time)
- one where y falls (moving south over time).

Basic Idea

So we end up with a diagram that looks something like this:



All points in a given quadrant move (qualitatively) in the same direction.

Now we can use logic to figure out possible paths.

Movement of x

From $\dot{x} = A - ax + by$:

$$\dot{x} = 0 \Rightarrow y = -\frac{A}{b} + \frac{a}{b}x \tag{24}$$

This divides the (x, y) plan into two regions:

- From : to the right of the boundary $x \downarrow$
- Mark this with arrows in the phase diagram.

Movement of y

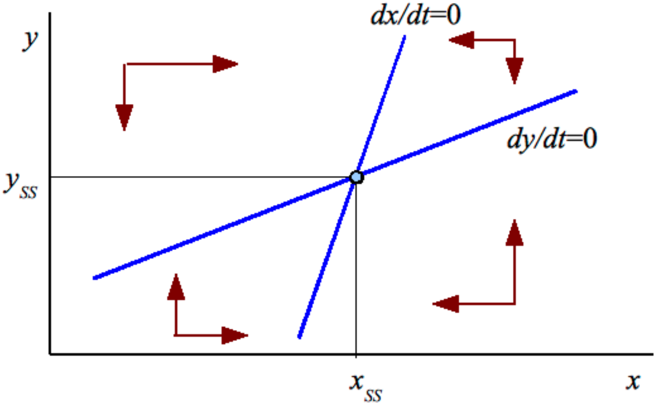
From $\dot{y} = B + cx - dy$:

$$\dot{y} = 0 \Rightarrow y = \frac{B}{d} + \frac{c}{d}x \qquad (25)$$

This divides the (x, y) plane into two regions as well:

- to the right of the boundary $y \uparrow$
- Mark this with arrows in the phase diagram.

Phase Diagram



Recall: $\dot{x} = A - ax + by$. $\dot{y} = B + cx - dy$.

Think about possible paths

For each quadrant, determine whether the path can leave the quadrant.

In this example: the answer is

- "no" for two quadrants.
- "yes" for two quadrants.

Now think about possible paths...

Conclude that the steady state is globally stable.

Applications

Unified growth theory:

- Galor (2000) studies transition from Malthusian stagnation to industrialization using a sequence of phase diagrams
- Galor (2005)

Models of human capital accumulation over the life-cycle:

- Heckman (1976)

Phase Diagram: Growth Model

Movement of c:

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \tag{26}$$

The $\dot{c} = 0$ locus is characterized by

$$f'(k^*) = \rho + \delta \tag{27}$$

A vertical line in the (k, c) plane.

Higher k implies lower $\dot{c}$

$c \downarrow$ to the right of the $\dot{c} = 0$ locus.

Movement of k

$$\dot{k} = f(k) - (n+\delta)k - c \tag{28}$$

The $\dot{k} = 0$ locus is hump-shaped:

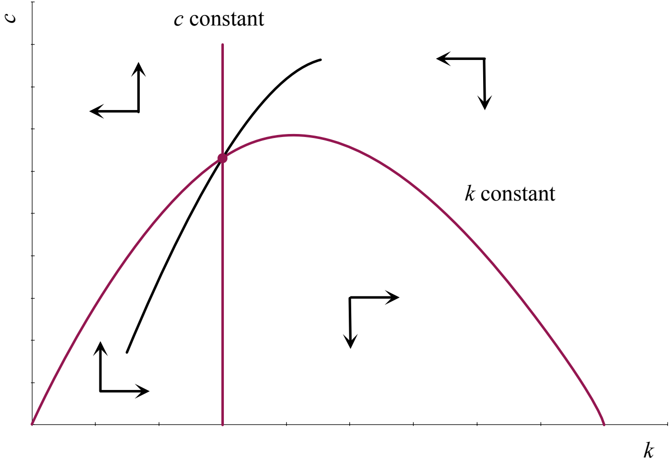
$$c = f(k) - (n+\delta)k \tag{29}$$

with a maximum at

$$f'(k^*) = n + \delta \tag{30}$$

Higher c implies lower $\dot{k}$.
k falls over time above the $\dot{k} = 0$ locus.

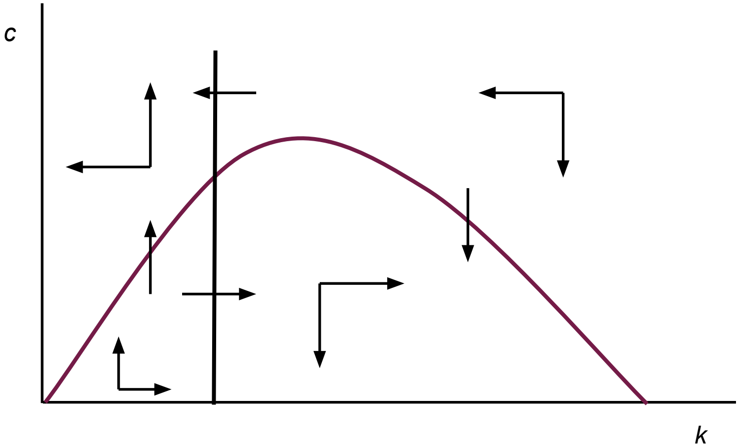
Phase Diagram



Since $\rho - n > 0$, the $\dot{c} = 0$ locus lies to the left of the peak of the $\dot{k} = 0$ locus.

The steady state is located at the intersection of the two curves.

Dynamics: Possible Paths



Ruling out the "north-west" path

$g(c)$ rises over time as $k \to 0$.

Eventually, this violates feasibility.

Ruling out the "south-east" path

Properties of that path:

$$c \to 0 \implies k \to k_{max} > k_{GR} \tag{31}$$

and therefore $f'(k_{max}) - \delta < n$.

This is exactly the kind of path that the TVC rules out.

Note:

- Even though $g(c)$ is strictly negative, $\dot{c} \to 0$. Therefore c does not turn negative.
- Any such path asymptotes towards $c = 0$ and $k = k_{max}$.

Ruling out the "south-east" path

Transversality

$$\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)k_t = 0 \tag{32}$$

requires

$$g\left(e^{-(\rho-n)t}u'(c)\right) = g(u') - (\rho-n) < 0 \tag{33}$$

as $t\to\infty$.

Euler:

$$g(u') = \rho+\delta-f'(k) \tag{34}$$

Higher k implies lower f' and higher $g(u')$.

Any $k > k_{GR}$ violates TVC because

$$g(u') - (\rho-n) = -\left[f'(k)-\delta-n\right] > 0 \tag{35}$$

Dynamics: Saddle-path Stability

Only one value of c avoids moving into “forbidden” regions for given k.

For this c, the economy converges to the steady state.

Such a system is called "**saddle-path stable**."

Details

Summary

The growth model in continuous time behaves like the one in discrete time (no surprise here):

1. The allocation is determined by Euler equation and resource constraint.
   Boundary conditions also look like in discrete time.
2. There is a unique, globally stable steady state.
   It satisfies "interest rate" = discount rate.
3. In continuous time, we can use a phase diagram to study the entire path.

Reading

- Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- Barro and Sala-i Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- Blanchard and Fischer (1989), ch. 2
- Romer (2011), ch. 2A
- Phase diagram: Barro and Sala-i Martin (1995), ch. 2.6

Technical notes: Unique saddle path

Theorem

Take as given
$\dot{x}(t) = G[x(t)]$ *with initial value* $x(0)$ *given, where* G *is continuously differentiable.*
The steady state is $G(x^*) = 0$. *Define* $A = DG(x^*)$.
Suppose that m *eigenvalues of* A *have negative real parts while* $n - m$ *have positive real parts.*
Then there exists an m *dimensional manifold in the neighborhood around the steady state such that starting from any* $x(0)$ *in that manifold a unique* $x(t) \to x^*$.

See Acemoglu (2009), Theorem 7.15.

What this says in words

Suppose we have a system of $n = 2$ differential equations (in c and k).

The local dynamics around the steady state can be approximated by a linear differential equation with matrix A.

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = A \begin{bmatrix} k \\ c \end{bmatrix} \tag{36}$$

If that matrix has $m = 1$ negative eigenvalues, then **locally** around the steady state there is a line (dimension $m = 1$) of points (c, k) that converge to the steady state.

- This is the saddle path.

Other points could, in principle, converge as well, but we can rule that out as above.

Application to the growth model

First, establish that the saddle path is locally unique.

Start from a linear approximation to the two differential equations:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*)\sigma & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix} \tag{37}$$

Details:

$$\dot{k} = f(k) - (n+\delta)k - c \tag{38}$$

$$= f'(k - k^*) - (n+\delta) - (c - c^*) \tag{39}$$

and

$$\dot{c} = c[f(k) - \delta - \rho]/\sigma \tag{40}$$

$$= (c - c^*)\underbrace{[f(k^*) - \delta - \rho]/\sigma}_{=0} + c^* \frac{f''(k^*)}{\sigma}(k - k^*) \tag{41}$$

Eigenvalues I

The eigenvalues λ solve $Ax = \lambda x$. In this case:

$$\begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*)/\sigma & 0 \end{bmatrix} x = \lambda x \tag{42}$$

Also $\det(A - I\lambda)x = 0$. In this case:

$$\det \begin{bmatrix} f'(k^*) - n - \delta - \lambda & -1 \\ c^* f''(k^*)/\sigma & 0 - \lambda \end{bmatrix} = 0 \tag{43}$$

$$\det(A - I\lambda) = -\left[f'(k^*) - \delta - n - \lambda\right]\lambda + c^* f''(k^*)/\sigma \tag{44}$$

$$\lambda^2 - \left(f' - \delta - n\right)\lambda + c^* f''/\sigma = 0 \tag{45}$$

Eigenvalues II

Apply

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{46}$$

and obtain

$$\lambda = \left\{ f'(k^*) - n - \delta \pm \sqrt{(f' - n - \delta)^2 - 4c^* f''/\sigma} \right\}/2 \tag{47}$$

Since $\sqrt{(f' - n - \delta)^2 - 4c^* f''/\sigma} > f' - n - \delta$, there is exactly one negative eigenvalue.

Therefore: in a neighborhood of the steady state, the saddle path is unique.

Application to the growth model

How do we know that the saddle is globally unique?

Define one saddle path that converges.

Take a point not on it. We know:

1. The path cannot reach or cross the saddle path in finite time.
2. The path cannot asymptote to the saddle path because that would get into a neighborhood of the steady state where the saddle is unique.
3. Therefore, the path cannot converge to the steady state.

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