Comparative Dynamics

Prof. Lutz Hendricks

Econ720

July 29, 2019
We use phase diagrams to uncover the dynamic response to shocks.

We study tax changes in a growth model.
Model

The **household** solves

\[
\max \int_0^\infty e^{-\rho t} u(c_t) \, dt
\] (1)

subject to

\[
\dot{k}_t = r_t k_t + w_t - c_t - \tau_t
\] (2)

and \( k_0 \) given.

**Firms** produce output using \( F(K, L) \).

The **government** uses the tax revenue to finance government spending: \( G_t = \tau_t \).
Competitive Equilibrium

A competitive equilibrium consists of functions $c(t), k(t), \tau(t), w(t), r(t)$ that satisfy:

1. Household: Budget constraint and
   \[ g(c) = \frac{r - \rho}{\sigma} \]  
   \[ (3) \]

2. Firms:
   \[ r = f'(k) - \delta \]  
   \[ (4) \]
   \[ w = f(k) - f'(k)k \]  
   \[ (5) \]

3. Government:
   \[ \tau = G \]  
   \[ (6) \]

4. Market clearing:
   \[ \dot{k} = f(k) - \delta k - c - G \]  
   \[ (7) \]
Phase Diagram

The only change relative to the model without government:
Permanent Tax Increase

Consider a permanent, unannounced increase in $G$. In the phase diagram

- $\dot{k} = 0$ locus shifts down by $\Delta G$.
- $k_{ss}$ remains unchanged because the $\dot{c} = 0$ locus does not shift.

Dynamics: $c_{ss}$ drops to the new saddle path, then moves along it.

- How do I know this is true?

An interesting long-run result: full crowding out of consumption ($\Delta c_{ss} = -\Delta G$).
Consider a *temporary*, unannounced increase in $G$.

- $G_t = G^* + \Delta G$ for $0 \leq t \leq T$, but $G_t = G^*$ for $t > T$.

To find the dynamics, we work backwards.

Changes occur at $t = 0$ and $t = T$.

Step 1: $t = T$. What happens?
Temporary Tax Increase

Step 2: $0 < t < T$:

- The phase diagram with taxes applies.
- But the economy is not on the saddle path (why not?).
- What is the right terminal condition for $k(T)$?
Temporary Tax Increase

Step 3: \( t = 0 \)

- The \( \dot{k} = 0 \) locus shifts down.
- Is \( c_0 \) on the saddle path?

Consider \( k_0 = k_{ss} \). What paths are feasible?
Temporary Tax Increase

$k$ constant
Temporary Tax Increase

Consider $k_0 < k_{ss}$. 
Consider a surprise tax cut that is announced to take place at date $T$.
At $t = 0$ the news arrives that taxes remain high until $t = T$, but then fall permanently.
Again, we work backwards.
Changes occur at $t = T$ and $t = 0$. 
Announced Tax Cut
Summary

To study the dynamic effects of shocks:

1. Find the phase diagram with and without shock.
2. Find the dates at which changes occur:
   2.1 when the shock hits: phase diagram changes 
      the control (typically) does not jump
   2.2 when new info arrives: agents reoptimize 
      the control jumps
3. Work backwards, starting at the last date at which a change occurs
Phase Diagram for a Simple Human Capital Model
A Human Capital Model

We study the decision of a household how much human capital to accumulate.

This example illustrates two complications:

1. finite horizons
2. binding inequality constraints.
Household problem

The household maximizes

$$\int_{0}^{T} e^{-\rho t} u(c(t)) \, dt$$

subject to the budget constraint

$$c(t) = w(t) h(t) [1 - \tau(t) v(t)]$$

the human capital technology

$$\dot{h}(t) = v(t) - \delta h(t)$$

and \(v \geq 0\).

For simplicity, assume that \(v \leq 1\) never binds.
Human capital acquired early is more valuable for two reasons:
1. it lives longer (date $T$ is farther off);
2. its payoffs are discounted by less.

We expect the optimal path for $v(t)$ to be falling over time.
When close to $T$, we expect $v(t) \geq 0$ to bind.
Hamiltonian

\[ H = u(wh[1 - \tau v]) + \lambda [v - \delta h] \]  \hspace{1cm} (11)

First-order conditions

\[ u'(c)wh\tau \geq \lambda \]  \hspace{1cm} (12)

with equality if \( v > 0 \) and

\[ \dot{\lambda} = \rho \lambda - u'(c)w(1 - \tau v) + \lambda \delta \]  \hspace{1cm} (13)
Summary

The solution to the household problem consists of functions $(c, h, v, \lambda)$ that solve

1. The first-order conditions

$$u'(c) \tau \geq \lambda$$

$$\dot{\lambda} = (\rho + \delta)\lambda - u'(c)w(1 - \tau v)$$

with equality if $v > 0$.

2. The budget constraint

$$c(t) = w(t)h(t)[1 - \tau(t)v(t)]$$

3. The law of motion

$$\dot{h}(t) = v(t) - \delta h(t)$$

4. The boundary conditions: $h_0$ given and $\lambda_T = 0$. 
Log utility

Assume $u(c) = \ln(c)$

Consider two regions of the phase diagram:

1. $v = 0$
2. $v > 0$
Region $v = 0$

$$wh\tau \geq \lambda c \quad (18)$$

$$\dot{\lambda} = (\rho + \delta) \lambda - w/c \quad (19)$$

$$c = wh \quad (20)$$

$$\dot{h} = -\delta h \quad (21)$$

Simplify:

$$\tau \geq \lambda \quad (22)$$

$$\dot{\lambda} = (\rho + \delta) \lambda - 1/h \quad (23)$$

$$\dot{h} = -\delta h \quad (24)$$

Terminal condition: $\lambda_T = 0$
Region \( v = 0 \)

- The shadow price \( \lambda \) is not large enough to cover the opportunity cost \( \tau \).
- The household does not invest in human capital.
- The laws of motion are:
  
  \[
  \begin{align*}
  \dot{\lambda} &= (\rho + \delta)\lambda - 1/h \\
  \dot{h} &= -\delta h
  \end{align*}
  \]

- \( \lambda \uparrow \Rightarrow \dot{\lambda} \uparrow \).
- \( h \uparrow \Rightarrow \dot{\lambda} \uparrow \) and \( \dot{h} \downarrow \).
- Hence, \( h(t) = h(t_0) e^{-\delta(t-t_0)} \), where \( t_0 \) is any date at which the economy is inside the region.
Phase Diagram: Region $\nu = 0$
Region \( \nu > 0 \)

\[
\frac{w h \tau}{c} = \lambda \tag{25}
\]

\[
\dot{\lambda} = (\rho + \delta) \lambda - \frac{w (1 - \tau \nu)}{c} \tag{26}
\]

\[
c = \nu h (1 - \tau \nu) \tag{27}
\]

\[
\dot{h} = \nu - \delta h \tag{28}
\]

Simplify:

\[
\lambda = \frac{\tau}{1 - \tau \nu}
\]

\[
\dot{\lambda} = (\rho + \delta) \lambda - \frac{1}{h}
\]

\[
\dot{h} = \nu - \delta h
\]
Region $\nu > 0$

The first-order condition for $\nu$ holds with equality:

$$\lambda (1 - \tau \nu) = \tau$$

or

$$\nu = 1/\tau - 1/\lambda$$  \hspace{1cm} (29)

Substitute $\nu$ out of the law of motion:

$$\dot{h} = 1/\tau - 1/\lambda - \delta h$$  \hspace{1cm} (30)

Keep

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h$$  \hspace{1cm} (31)
Region $v > 0$

- In this region, the shadow price of human capital ($\lambda$) equals the opportunity cost.
  \[ \lambda > \tau \]

- $\dot{h} = \frac{1}{\tau} - \frac{1}{\lambda} - \delta h = 0$
  is upward sloping and starts at $\lambda = \tau$.

- $\dot{\lambda} = (\rho + \delta)\lambda - \frac{1}{h} = 0$
  is a downward sloping hyperbola (as in region $v = 0$).

- $h \uparrow$ or $\lambda \downarrow \Rightarrow \dot{h} \downarrow$. 
Phase Diagram
Steady State

- Assume that $w$ and $\tau$ are constant over time and that $T = \infty$.
- Then $h$ and $v$ converge to stationary levels, $h_{ss}$ and $v_{ss}$.
- We next determine those levels.
- $\dot{\lambda} = 0$ implies
  \[
  (\rho + \delta) h \lambda = (\rho + \delta) h \frac{\tau}{1 - \tau v} = 1
  \]
  (32)

- $\dot{h} = 0$ implies
  \[
  v = \delta h
  \]
  (33)

- Combine both
  \[
  h_{ss} = [\tau (\rho + 2 \delta)]^{-1}
  \]
  (34)
Steady State

It follows that

\[ v_{ss} = \delta \bar{h}_{ss} = \frac{\delta}{\tau[\rho + 2 \delta]} \]

\[ c_{ss} = \frac{(\rho + \delta)w}{(\rho + 2 \delta)^2 \tau} \]

\[ \lambda_{ss} = u' \left( \frac{(\rho + \delta)w}{(\rho + 2 \delta)^2 \tau} \right) \frac{w}{\rho + 2 \delta} \]
The phase diagram has two regions: \( v = 0 \) and \( v > 0 \).

The region boundary occurs when the household just hits the constraint \( v \geq 0 \): at \( \lambda = \tau \).

For \( \lambda > \tau \): \( v > 0 \).

For \( \lambda \leq \tau \): \( v = 0 \).
Phase Diagram
Dynamics

- Any path must end with \( \lambda_T = 0 \) exactly at date \( T \).
- It follows that the shaded region must never be entered.
- What happens as the steady state is approached with \( v > 0 \)?
  - Since all the laws of motion are continuous, \( \dot{h} \rightarrow 0 \) and \( \dot{\lambda} \rightarrow 0 \).
  - The steady state can never be reached.
  - But the economy can spend an arbitrarily long time arbitrarily close to the steady state.
Dynamics

- First consider $h_0 < h_{ss}$.
- $\lambda$ depends on the horizon $T$.
- Short $T$: $\lambda$ is low.
  - Start in region $v = 0$
  - move south-west until $\lambda_T = 0$. 
To prove this, solve the two differential equations.

\[ h(t) = h(t_0)e^{-\delta t} \]

Substitute this into the law of motion for \( \lambda \) to obtain

\[ \dot{\lambda} = (\rho + \delta) \lambda - e^{\delta t}/h(t_0) \quad (35) \]

The solution to this differential equation is

\[ \lambda(t) = e^{(\rho+\delta)(t-t_0)} \left[ \lambda(t_0) - \frac{\rho}{h(t_0)} \left\{ 1 - e^{-\rho(t-t_0)} \right\} \right] \]

Imposing the boundary condition \( \lambda(T) = 0 \) implies

\[ \lambda(t_0)h(t_0) = \rho \left\{ 1 - e^{-\rho(T-t_0)} \right\}. \]

For long \( T \): \( \lambda(t_0) \rightarrow \rho/h(t_0) \) (unless the region \( v = 0 \) is left).

But for a short \( T \), \( \lambda(t_0) \rightarrow 0 \).
Dynamics

- Case: $h_0 < h_{ss}$ and long $T$.
- Initially $v > 0$ and the economy moves south until it crosses into the $v = 0$ region.
- As $T \to \infty$ something bizarre happens:
  - the economy approaches the steady state without ever reaching it.
  - It comes arbitrarily close and stays arbitrarily close for an arbitrarily long time.
  - But when the terminal date comes sufficiently close it leaves the steady state and moves south-west to reach $\lambda_T = 0$. 
Dynamics

- Case $h_0 > h_{ss}$.
- Investment is never large enough to increase $h$.
- The economy may move straight south-west if $T$ is short or it may move towards the steady state, similar to the case where $h_0 < h_{ss}$.
Reading

- Acemoglu, Introduction to modern economic growth, ch. 8.7.