

Comparative Dynamics

Prof. Lutz Hendricks

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Comparative Dynamics

- ▶ We use phase diagrams to uncover the dynamic response to shocks.
- ▶ We study tax changes in a growth model.

Model

The **household** solves

$$\max \int_0^{\infty} e^{-\rho t} u(c_t) dt \quad (1)$$

subject to

$$\dot{k}_t = r_t k_t + w_t - c_t - \tau_t \quad (2)$$

and k_0 given.

Firms produce output using $F(K, L)$.

The **government** uses the tax revenue to finance government spending: $G_t = \tau_t$.

Competitive Equilibrium

A competitive equilibrium consists of functions $c(t), k(t), \tau(t), w(t), r(t)$ that satisfy:

1. Household: Budget constraint and

$$g(c) = \frac{r - \rho}{\sigma} \quad (3)$$

2. Firms:

$$r = f'(k) - \delta \quad (4)$$

$$w = f(k) - f'(k)k \quad (5)$$

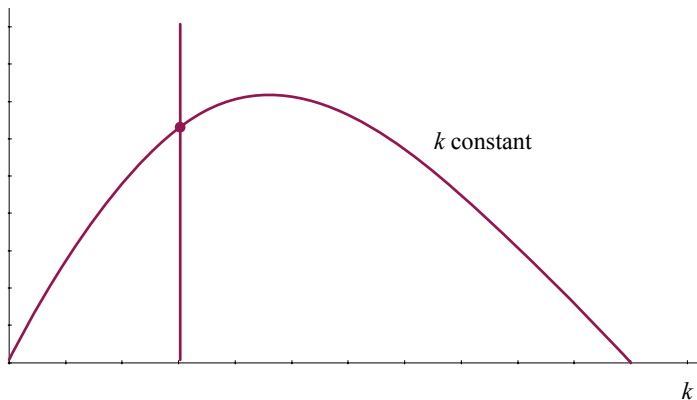
3. Government:

$$\tau = G \quad (6)$$

4. Market clearing:

$$\dot{k} = f(k) - \delta k - c - G \quad (7)$$

Phase Diagram



The only change relative to the model without government:

Permanent Tax Increase

Consider a permanent, unannounced increase in G .

In the phase diagram

- ▶ $\dot{k} = 0$ locus shifts down by ΔG .
- ▶ k_{ss} remains unchanged because the $\dot{c} = 0$ locus does not shift.

Dynamics: c_{ss} drops to the new saddle path, then moves along it.

- ▶ How do I know this is true?

An interesting long-run result: full crowding out of consumption ($\Delta c_{ss} = -\Delta G$).

Temporary Tax Increase

- ▶ Consider a *temporary*, unannounced increase in G .
 - ▶ $G_t = G^* + \Delta G$ for $0 \leq t \leq T$, but $G_t = G^*$ for $t > T$.
- ▶ To find the dynamics, we work backwards.
- ▶ Changes occur at $t = 0$ and $t = T$.

Step 1: $t = T$. What happens?

Temporary Tax Increase

Step 2: $0 < t < T$:

- ▶ The phase diagram with taxes applies.
- ▶ But the economy is not on the saddle path (why not?).
- ▶ What is the right terminal condition for $k(T)$?

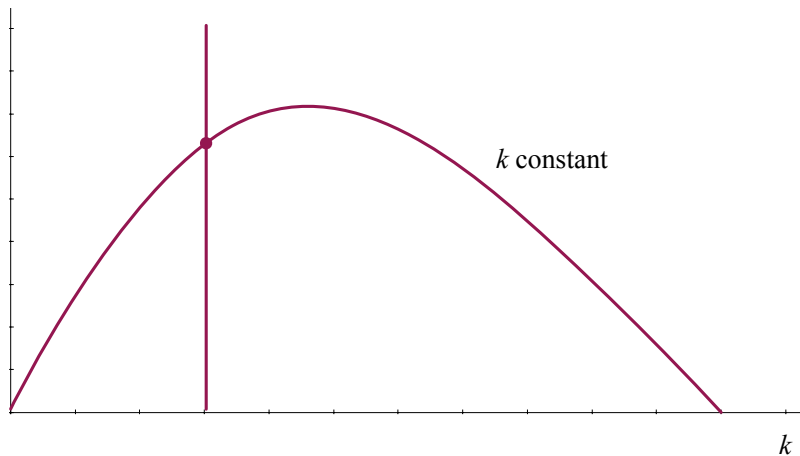
Temporary Tax Increase

Step 3: $t = 0$

- ▶ The $\dot{k} = 0$ locus shifts down.
- ▶ Is c_0 on the saddle path?

Consider $k_0 = k_{ss}$. What paths are feasible?

Temporary Tax Increase



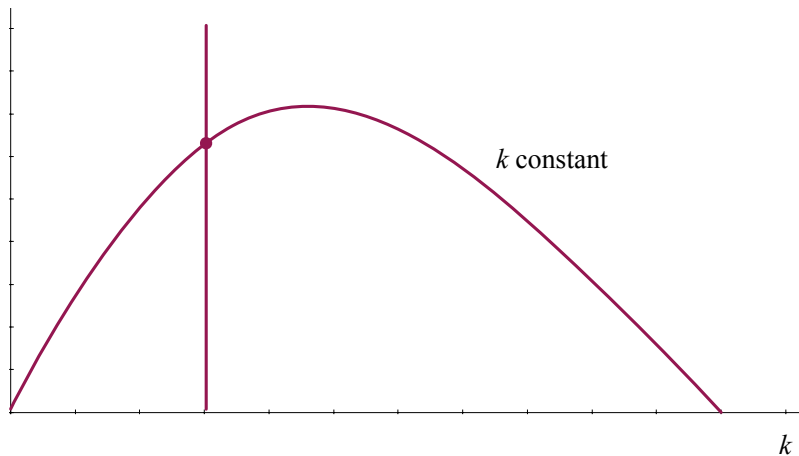
Temporary Tax Increase

Consider $k_0 < k_{ss}$.

Announced Tax Cut

- ▶ Consider a surprise tax cut that is announced to take place at date T .
- ▶ At $t = 0$ the news arrives that taxes remain high until $t = T$, but then fall permanently.
- ▶ Again, we work backwards.
- ▶ Changes occur at $t = T$ and $t = 0$.

Announced Tax Cut



Summary

To study the dynamic effects of shocks:

1. Find the phase diagram with and without shock.
2. Find the dates at which changes occur:
 - 2.1 when the shock hits: phase diagram changes
the control (typically) does not jump
 - 2.2 when new info arrives: agents reoptimize
the control jumps
3. Work backwards, starting at the last date at which a change occurs

Phase Diagram for a Simple Human Capital Model

A Human Capital Model

We study the decision of a household how much human capital to accumulate.

This example illustrates two complications:

1. finite horizons
2. binding inequality constraints.

Household problem

The household maximizes

$$\int_0^T e^{-\rho t} u(c(t)) dt \quad (8)$$

subject to the budget constraint

$$c(t) = w(t)h(t)[1 - \tau(t)v(t)] \quad (9)$$

the human capital technology

$$\dot{h}(t) = v(t) - \delta h(t) \quad (10)$$

and $v \geq 0$.

For simplicity, assume that $v \leq 1$ never binds.

Household: Intuition

- ▶ Human capital acquired early is more valuable for two reasons:
 1. it lives longer (date T is farther off);
 2. its payoffs are discounted by less.
- ▶ We expect the optimal path for $v(t)$ to be falling over time.
- ▶ When close to T , we expect $v(t) \geq 0$ to bind.

Hamiltonian

$$H = u(wh[1 - \tau v]) + \lambda [v - \delta h] \quad (11)$$

First-order conditions

$$u'(c)wh\tau \geq \lambda \quad (12)$$

with equality if $v > 0$ and

$$\dot{\lambda} = \rho\lambda - u'(c)w(1 - \tau v) + \lambda\delta \quad (13)$$

Summary

The solution to the household problem consists of functions (c, h, v, λ) that solve

1. The first-order conditions

$$u'(c)wh\tau \geq \lambda \quad (14)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - u'(c)w(1 - \tau v) \quad (15)$$

with equality if $v > 0$.

2. The budget constraint

$$c(t) = w(t)h(t)[1 - \tau(t)v(t)] \quad (16)$$

3. The law of motion

$$\dot{h}(t) = v(t) - \delta h(t) \quad (17)$$

4. The boundary conditions: h_0 given and $\lambda_T = 0$.

Log utility

Assume $u(c) = \ln(c)$

Consider two regions of the phase diagram:

1. $v = 0$
2. $v > 0$

Region $v = 0$

$$wh\tau \geq \lambda c \quad (18)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - w/c \quad (19)$$

$$c = wh \quad (20)$$

$$\dot{h} = -\delta h \quad (21)$$

Simplify:

$$\tau \geq \lambda \quad (22)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h \quad (23)$$

$$\dot{h} = -\delta h \quad (24)$$

Terminal condition: $\lambda_T = 0$

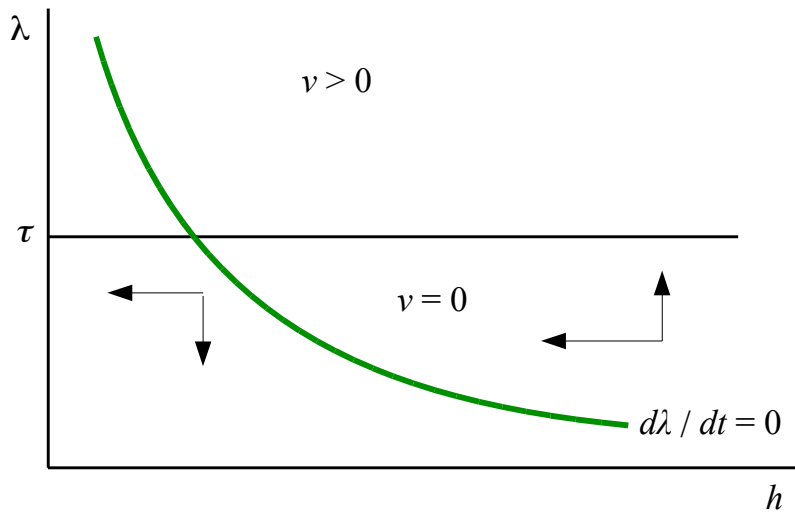
Region $v = 0$

- ▶ The shadow price λ is not large enough to cover the opportunity cost τ .
- ▶ The household does not invest in human capital.
- ▶ The laws of motion are:

$$\begin{aligned}\dot{\lambda} &= (\rho + \delta)\lambda - 1/h \\ \dot{h} &= -\delta h\end{aligned}$$

- ▶ $\lambda \uparrow \Rightarrow \dot{\lambda} \uparrow$.
- ▶ $h \uparrow \Rightarrow \dot{\lambda} \uparrow$ and $\dot{h} \downarrow$.
- ▶ Hence, $h(t) = h(t_0)e^{-\delta(t-t_0)}$, where t_0 is any date at which the economy is inside the region.

Phase Diagram: Region $v = 0$



Region $v > 0$

$$\frac{wh\tau}{c} = \lambda \quad (25)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - \frac{w(1 - \tau v)}{c} \quad (26)$$

$$c = wh(1 - \tau v) \quad (27)$$

$$\dot{h} = v - \delta h \quad (28)$$

Simplify:

$$\lambda = \frac{\tau}{1 - \tau v}$$

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h$$

$$\dot{h} = v - \delta h$$

Region $\nu > 0$

The first-order condition for ν holds with equality:

$$\lambda(1 - \tau\nu) = \tau$$

or

$$\nu = 1/\tau - 1/\lambda \quad (29)$$

Substitute ν out of the law of motion:

$$\dot{h} = 1/\tau - 1/\lambda - \delta h \quad (30)$$

Keep

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h \quad (31)$$

Region $v > 0$

- ▶ In this region, the shadow price of human capital (λ) equals the opportunity cost.

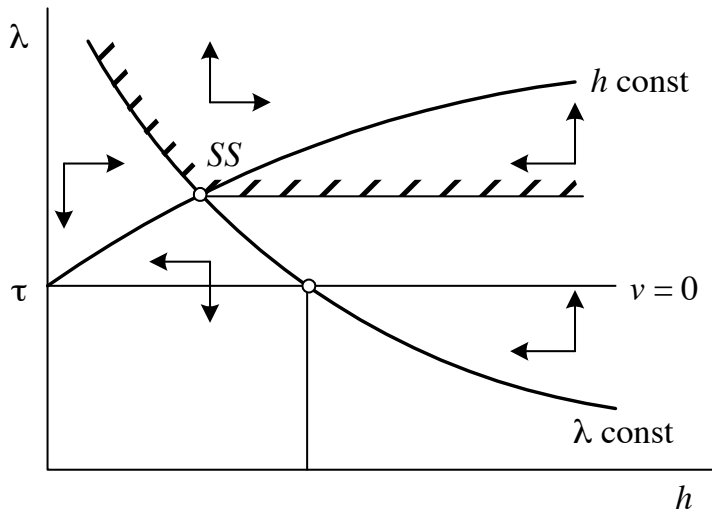
$$\lambda > \tau$$

- ▶ $\dot{h} = 1/\tau - 1/\lambda - \delta h = 0$
is upward sloping and starts at $\lambda = \tau$.

- ▶ $\dot{\lambda} = (\rho + \delta)\lambda - 1/h = 0$
is a downward sloping hyperbola (as in region $v = 0$).

- ▶ $h \uparrow$ or $\lambda \downarrow \Rightarrow \dot{h} \downarrow$.

Phase Diagram



Steady State

- ▶ Assume that w and τ are constant over time and that $T = \infty$.
- ▶ Then h and v converge to stationary levels, h_{ss} and v_{ss} .
- ▶ We next determine those levels.
- ▶ $\dot{\lambda} = 0$ implies

$$(\rho + \delta)h\lambda = (\rho + \delta)h\frac{\tau}{1 - \tau v} = 1 \quad (32)$$

- ▶ $\dot{h} = 0$ implies

$$v = \delta h \quad (33)$$

- ▶ Combine both

$$h_{ss} = [\tau(\rho + 2\delta)]^{-1} \quad (34)$$

Steady State

It follows that

$$v_{ss} = \delta h_{ss} = \frac{\delta}{\tau[\rho + 2\delta]}$$

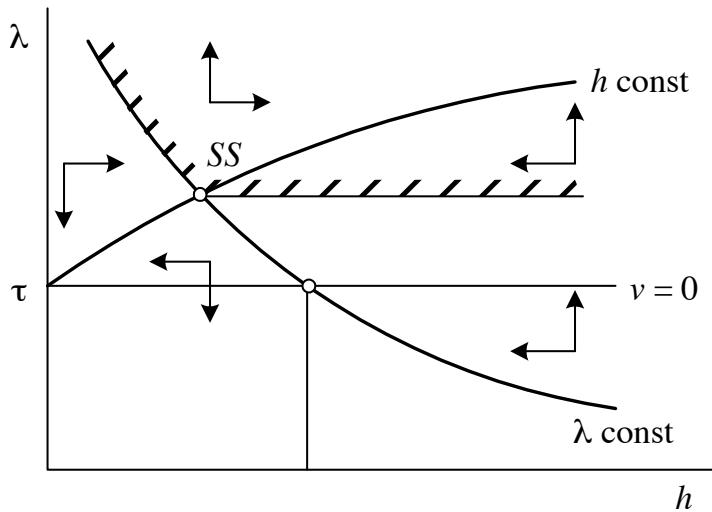
$$c_{ss} = \frac{(\rho + \delta)w}{(\rho + 2\delta)^2 \tau}$$

$$\lambda_{ss} = u' \left(\frac{(\rho + \delta)w}{(\rho + 2\delta)^2 \tau} \right) \frac{w}{\rho + 2\delta}$$

Phase Diagram

- ▶ The phase diagram has two regions: $v = 0$ and $v > 0$.
- ▶ The region boundary occurs when the household just hits the constraint $v \geq 0$: at $\lambda = \tau$.
- ▶ For $\lambda > \tau$: $v > 0$.
- ▶ For $\lambda \leq \tau$: $v = 0$.

Phase Diagram



Dynamics

- ▶ Any path must end with $\lambda_T = 0$ exactly at date T .
- ▶ It follows that the shaded region must never be entered.
- ▶ What happens as the steady state is approached with $v > 0$?
 - ▶ Since all the laws of motion are continuous, $\dot{h} \rightarrow 0$ and $\dot{\lambda} \rightarrow 0$.
 - ▶ The steady state can never be reached.
 - ▶ But the economy can spend an arbitrarily long time arbitrarily close to the steady state.

Dynamics

- ▶ First consider $h_0 < h_{ss}$.
- ▶ λ depends on the horizon T .
- ▶ Short T : λ is low.
 - ▶ Start in region $v = 0$
 - ▶ move south-west until $\lambda_T = 0$.

Dynamics

- ▶ To prove this, solve the two differential equations.
- ▶ $h(t) = h(t_0) e^{-\delta t}$. Substitute this into the law of motion for λ to obtain

$$\dot{\lambda} = (\rho + \delta)\lambda - e^{\delta t}/h(t_0) \quad (35)$$

- ▶ The solution to this differential equation is

$$\lambda(t) = e^{(\rho+\delta)(t-t_0)} \left[\lambda(t_0) - \frac{\rho}{h(t_0)} \left\{ 1 - e^{-\rho(t-t_0)} \right\} \right]$$

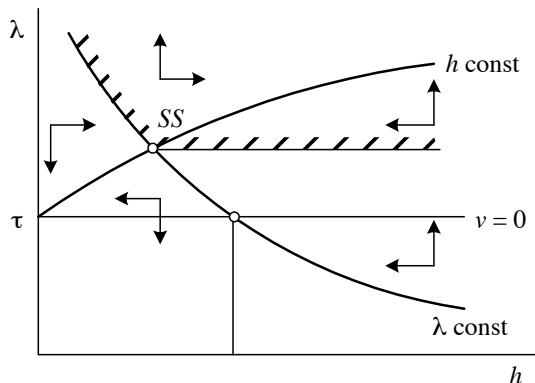
- ▶ Imposing the boundary condition $\lambda(T) = 0$ implies $\lambda(t_0)h(t_0) = \rho \{ 1 - e^{-\rho(T-t_0)} \}$.
- ▶ For long T : $\lambda(t_0) \rightarrow \rho/h(t_0)$ (unless the region $v = 0$ is left).
- ▶ But for a short T , $\lambda(t_0) \rightarrow 0$.

Dynamics

- ▶ Case: $h_0 < h_{ss}$ and long T .
- ▶ Initially $v > 0$ and the economy moves south until it crosses into the $v = 0$ region.
- ▶ As $T \rightarrow \infty$ something bizarre happens:
 - ▶ the economy approaches the steady state without ever reaching it.
 - ▶ It comes arbitrarily close and stays arbitrarily close for an arbitrarily long time.
 - ▶ But when the terminal date comes sufficiently close it leaves the steady state and moves south-west to reach $\lambda_T = 0$.

Dynamics

- ▶ Case $h_0 > h_{SS}$.
- ▶ Investment is never large enough to increase h .
- ▶ The economy may move straight south-west if T is short or it may move towards the steady state, similar to the case where $h_0 < h_{SS}$.



Reading

- ▶ Acemoglu, Introduction to modern economic growth, ch. 8.7.
- ▶ Hendricks, Lutz (2004). "Taxation and Human Capital Accumulation." *Macroeconomic Dynamics* 8(3): 310-334.
- ▶ Sheshinski, Eytan (1968), "On the Individual's Lifetime Allocation Between Education and Work," *Metroeconomica*, 20(1), 42-9.