# Optimal Control

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# Topics

Optimal control is a method for solving dynamic optimization problems in continuous time.

# Example: Growth Model

A household chooses optimal consumption to

$$\max \int_0^T e^{-\rho t} u[c(t)] dt \tag{1}$$

subject to

$$\dot{k}(t) = rk(t) - c(t) \tag{2}$$
$$c(t) \in [0, \bar{c}] \tag{3}$$
$$k(0) = k_0, \text{given} \tag{4}$$
$$k(T) \geq 0 \tag{5}$$

# Generic Optimal control problem

Choose functions of time $c(t)$ and $k(t)$ so as to

$$\max \int_0^T v[k(t), c(t), t] dt \tag{6}$$

Constraints:

1. Law of motion of the **state** variable $k(t)$:
$$\dot{k}(t) = g[k(t), c(t), t] \tag{7}$$
2. Feasible set for **control** variable $c(t)$:
$$c(t) \in Y(t) \tag{8}$$
3. Boundary conditions, such as:
$$k(0) = k_0, \text{given} \tag{9}$$
$$k(T) \geq k_T \tag{10}$$

# Generic Optimal control problem

- $c$ and $k$ can be vectors.
- $Y(t)$ is a compact, nonempty set.
- $T$ could be infinite.
  - Then the boundary conditions change
- Important: the state cannot jump; the control can.
- Note that this looks exactly like the kind of problem that could be solved with Dynamic Programming in discrete time.

# A Recipe for Solving Optimal Control Problems

# A Recipe

Step1. Write down the ***Hamiltonian***

$$H(t) = v(k,c,t) + \mu(t)\underbrace{g(k,c,t)}_{\dot{k}(t)} \tag{11}$$

$\mu$ is essentially a Lagrange multiplier (called a **co-state**).

Intuition:

- similar to the dynamic program: current utility + continuation value (but not quite)
- $v(k,c,t)$: current utility
- $\mu(t)$: the marginal value of increasing $k$ for the future
- $g(k,c,t)$: captures how current actions affect future $k$

# A Recipe

Step 2. Derive the **first order conditions** which are **necessary** for an optimum:

$$\partial H/\partial c = 0 \tag{12}$$
$$\partial H/\partial k = -\dot{\mu} \tag{13}$$

Intuition below ...

# A Recipe

Step 3. Impose the **transversality** condition:

- for finite horizon:

$$\mu(T) = 0 \tag{14}$$

- for infinite horizon:

$$\lim_{t \to \infty} H(t) = 0 \tag{15}$$

This depends on the terminal condition (see below).

# A Recipe

Step 4. A **solution** is the a set of functions $[c(t), k(t), \mu(t)]$ which satisfy

- the FOCs
- the law of motion for the state
- the boundary / transversality conditions

# Intuition $\partial H / \partial c = 0$

Maximize Hamiltonian w.r.to control.

Implies

$$v_c + \mu g_c = 0 \tag{16}$$

$v_c(k, c, t)$ picks up current marginal utility of $c$

$\mu(t)$ is marginal value of additional "future" $k$.

$\mu(t) g_c(k, c, t)$ picks up change in continuation value

(change in $\dot{k}$ times marginal value of future $k$)

# Intuition: $\partial H / \partial k = -\dot{\mu}$

Implies

$$v_k(k,c,t) + \mu g_k(k,c,t) = -\dot{\mu} \tag{17}$$

Think of this as

$$[\partial H / \partial k] / \mu = -\dot{\mu} / \mu \tag{18}$$

- $\dot{\mu}/\mu$ is the growth rate of marginal utility
- $[\partial H / \partial k] / \mu$ is like a rate of return (marginal value of $k$ now versus the future)
- if the rate of return is high, it is optimal to postpone consumption and let it grow
- then marginal utility declines over time

# Example: Growth Model

$$\max \int_0^\infty v(k,c,t)\,dt \to \max \int_0^\infty e^{-\rho t} u(c(t))\,dt \tag{19}$$

subject to

$$\begin{aligned}
\dot{k}(t) &= g(k,c,t) \equiv f(k(t)) - c(t) - \delta k(t) && (20)\\
c(t) &\in Y(t) \equiv [0, f(k_{max}) - \delta k_{max}] && (21)\\
& k(0) \text{ given} && (22)
\end{aligned}$$

For this to work, we need to bound $k \leq k_{max}$.

# Growth Model: Hamiltonian

$$H(k,c,\mu) = \underbrace{e^{-\rho t}u(c(t))}_{v(k,c,t)} + \mu(t)\underbrace{[f(k(t)) - c(t) - \delta k(t)]}_{\dot{k}} \qquad (23)$$

Necessary conditions:

$$\begin{aligned} H_c &= e^{-\rho t}u'(c) - \mu = 0 \\ H_k &= \mu\left[f'(k) - \delta\right] = -\dot{\mu} \end{aligned}$$

Interpretation:

- $\mu$ is indeed the marginal value of capital (the same as the marginal value of consumption)
- $-g(\mu) = f'(k) - \delta$: when the rate of return is high, marginal utility falls over time

# Substitute out the co-state

FOC imply:

$$
\begin{aligned}
g(\mu) &= \delta - f'(k) && (24)\\
&= \frac{d \ln\left(e^{-\rho t} u'(c_t)\right)}{dt} && (25)\\
&= \frac{d}{dt}\left[-\rho t + \ln u'(c_t)\right] && (26)\\
&= -\rho + \frac{u''(c_t)\,\dot{c}_t}{u'(c_t)} && (27)
\end{aligned}
$$

We could have used the growth rate rule:

$$
\begin{aligned}
g\left(e^{-\rho t} u'(c)\right) &= -\rho + g\left(u'(c)\right) && (28)\\
&= -\rho - \sigma(c)\, g(c) && (29)
\end{aligned}
$$

where $\sigma(c) = -u''/u' \times c$ is the elasticity of marginal utility w.r.to $c$

# Simplify

$$-g(\mu) = f'(k) - \delta = \rho + \sigma(c)\, g(c) \tag{30}$$

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma(c)}$$

Solution: $c_t, k_t$ that solve Euler equation and resource constraint, plus boundary conditions.

# Details

First order conditions are necessary, not sufficient.

They are necessary only if we **assume** that

1. a continuous, interior solution exists;
2. the objective function $v$ and the constraint function $g$ are continuously differentiable.

Acemoglu (2009), ch. 7, offers some insight into why the FOCs are necessary.

# Details

If there are multiple states and controls, simply write down one FOC for each separately:

$$
\begin{aligned}
\delta H / \delta c_i &= 0 \\
\partial H / \partial k_j &= -\dot{\mu}_j
\end{aligned}
$$

There is a large variety of cases depending on the length of the horizon (finite or infinite) and the kinds of boundary conditions.

- Each has its transversality condition (see Leonard and Van Long 1992).

# Sufficient conditions

First-order conditions are sufficient, if the programming problem is **concave**.

This can be checked in various ways.

# Sufficient conditions I

The objective function and the constraints are concave functions of the controls and the states.

- The co-state must be positive.

This condition is easy to check, but very stringent.

In the growth model:

- $u(c)$ is concave in $c$ (and, trivially, $k$)
- $f(k) - \delta k - c$ is concave in $c$ and $k$
- $\mu = u'(c) > 0$

# Sufficient Conditions II

(Mangasarian) First-order conditions are sufficient, if
the Hamiltonian is concave in controls and states,
where the co-state is evaluated at the optimal level (and held fixed).

This, too is very stringent.

Note: Conditions I $\implies$ II (the sum of two concave functions is concave).

# In the growth model

$$\partial H / \partial c = u'(c) - \mu$$
$$\partial H / \partial k = \mu [f'(k) - \delta]$$
$$\partial^2 H / \partial c^2 = u''(c) < 0$$
$$\partial^2 H / \partial k^2 = \mu f''(k) < 0$$
$$\partial^2 H / \partial c \partial k = 0$$

Therefore: weak joint concavity (because we know that $\mu > 0$)

# Sufficient Conditions III

Arrow and Kurz (1970)

- First-order conditions are sufficient, if the *maximized* Hamiltonian is concave in the states.
- If the maximized Hamiltonian is strictly concave in the states, the optimal path is unique.

Maximized Hamiltonian:

Substitute out the controls, so that the Hamiltonian is only a function of the states. (Keep the co-states).

This is less stringent and by far the most useful set of sufficient conditions.

# In the growth model

Optimal consumption obeys $u'(c) = \mu$ or $c = u'^{-1}(\mu)$

Maximized Hamiltonian:

$$\hat{H} = u\left(u'^{-1}(\mu)\right) + \mu\left[f(k) - \delta k - u'^{-1}(\mu)\right] \tag{31}$$

We have $\partial \hat{H}/\partial k > 0$ and $\partial^2 \hat{H}/\partial k^2 = \mu f''(k) < 0$.

$\hat{H}$ is strictly concave in $k$.

Necessary conditions yield a unique optimal path.

# Discounting: Current value Hamiltonian

Problems with discounting:

- Current utility depends on time only through an exponential discounting term $e^{-\rho t}$.

The generic discounted problem is

$$\max \int_0^T e^{-\rho t} v[k(t), c(t)] dt \tag{32}$$

subject to the same constraints as above.

# Shortcut

Discounted Hamiltonian (drop the discounting term):

$$H = v(k,c) + \mu g(k,c) \tag{33}$$

FOCs:

$$\partial H/\partial c = 0 \tag{34}$$

$$\partial H/\partial k = \underbrace{\mu(t)\rho}_{\text{added}} - \dot{\mu}(t) \tag{35}$$

and the TVC

$$\lim_{T\to\infty} e^{-\rho T}\mu(T)k(T) = 0 \tag{36}$$

# Deriving the Shortcut

Start from the standard recipe:

$$H(t) = e^{-\rho t} v(k,c) + \hat{\mu} g(k,c) \tag{37}$$

$$\frac{\partial H}{\partial c_t} = 0 \implies e^{-\rho t} v_c(k_{t,} c_t) = -\hat{\mu}_t g_c(k_t, c_t) \tag{38}$$

$$\frac{\partial H}{\partial k_t} = e^{-\rho t} v_k(k_t, c_t) + \hat{\mu}_t g_k(k_t, c_t) = -\dot{\hat{\mu}}_t \tag{39}$$

# Deriving the Shortcut

Let

$$\mu_t = e^{\rho t}\hat{\mu}_t \tag{40}$$

and multiply through by $e^{\rho t}$:

$$\frac{\partial H}{\partial c_t} = 0 \implies \underbrace{e^{\rho t}e^{-\rho t}}_{1} v_c(k_{t,}c_t) = -\underbrace{e^{\rho t}\hat{\mu}_t}_{\mu_t} g_c(k_t, c_t) \tag{41}$$

$$v_c(t) = -\mu_t g_c(t)$$

This is the standard FOC, but with $\mu$ instead of $\hat{\mu}$.

# Deriving the Shortcut

$$v_k(t) + e^{\rho t}\hat{\mu}_t g_k(t) = -e^{\rho t}\dot{\hat{\mu}}_t \quad (42)$$

Substitute out $\dot{\hat{\mu}}_t$ using

$$\dot{\mu}_t = \frac{de^{\rho t}\hat{\mu}_t}{dt} = \rho\mu_t + e^{\rho t}\dot{\hat{\mu}}_t$$

we have

$$v_k(t) + \mu_t g_k(t) = -\dot{\mu}_t + \rho\mu_t$$

This is the standard condition with an additional $\rho\mu$ term.

# Equality constraints

Equality constraints of the form

$$h[c(t), k(t), t] = 0 \tag{43}$$

are simply added to the Hamiltonian as in a Lagrangian problem:

$$H(t) = v(k, c, t) + \mu(t) g(k, c, t) + \lambda(t) h(k, c, t) \tag{44}$$

FOCs are unchanged:

$$\begin{aligned} \partial H / \partial c &= 0 \\ \partial H / \partial k &= -\dot{\mu} \end{aligned}$$

For inequality constraints:

$$h(c, k, t) \geq 0;\ \lambda h = 0 \tag{45}$$

# Transversality Conditions

# Finite horizon: Scrap value problems

The horizon is $T$.

The objective function assigns a scrap value to the terminal state variable: $e^{-\rho T}\phi(k(T))$:

$$\max \int_0^T e^{-\rho t} v[k(t), c(t), t] dt + e^{-\rho T}\phi(k(T)) \tag{46}$$

Hamiltonian and FOCs: unchanged.

The TVC is

$$\mu(T) = \phi'(k(T)) \tag{47}$$

Intuition: $\mu$ is the marginal value of the state $k$.

# Scrap value examples

1. Household with bequest motive

$$U = \int_0^T e^{-\rho t} u(c(t)) + e^{-\rho t} V(k_T) \tag{48}$$

   with $\dot{k} = w + rk - c$.

2. Maximizing the present value of earnings

$$Y = \int_0^T e^{-rt} w h(t) [1 - l(t)] \tag{49}$$

   subject to $\dot{h}(t) = Ah(t)^{\alpha} l(t)^{\beta} - \delta h(t)$
   Scrap value is $0$. TVC: $\mu(T) = 0$.

# Infinite horizon TVC

The finite horizon TVC with the boundary condition $k(T) \geq k_T$ is $\mu(T) = 0$.

- Intuition: capital has no value at the end of time.

But the infinite horizon boundary condition is NOT $\lim_{t \to \infty} \mu(t) = 0$.

The next example illustrates why.

# Infinite horizon TVC: Example

$$
\begin{aligned}
& \max \int_0^\infty [\ln(c(t)) - \ln(c^*)]\, dt \\
& \textit{subject to} \\
\dot{k}(t) &= k(t)^\alpha - c(t) - \delta k(t) \\
k(0) &= 1 \\
\lim_{t\to\infty} k(t) &\geq 0
\end{aligned}
$$

$c^*$ is the max steady state (golden rule) consumption.

No discounting - subtracting $c^*$ makes utility finite.

# Infinite horizon TVC

Hamiltonian

$$H(k,c,\lambda) = \ln c - \ln c^* + \lambda\left[k^\alpha - c - \delta k\right] \tag{50}$$

Necessary FOCs

$$\begin{aligned} H_c &= 1/c - \lambda = 0 && (51)\\ H_k &= \lambda\left[\alpha k^{\alpha-1} - \delta\right] = -\dot{\lambda} && (52)\end{aligned}$$

# Infinite horizon TVC

We show: $\lim_{t\to\infty} c(t) = c^*$ [why?]

Limiting steady state solves

$$\begin{aligned} \dot{\lambda}/\lambda &= \alpha k^{\alpha-1} - \delta = 0 \\ \dot{k} &= k^{\alpha} - 1/\lambda - \delta k = 0 \end{aligned}$$

Solution is the golden rule:

$$k^* = (\alpha/\delta)^{1/(1-\alpha)} \tag{53}$$

Verify that this max's steady state consumption.

# Infinite horizon TVC

Implications for the TVC...

$\lambda(t) = 1/c(t)$ implies $\lim_{t\to\infty} \lambda(t) = 1/c^*$.

Therefore, neither $\lambda(t)$ nor $\lambda(t)k(t)$ converge to 0.

The correct TVC:

$$\lim_{t\to\infty} H(t) = 0 \tag{54}$$

The only reason why the standard TVC does not work: there is **no discounting** in the example.

# Infinite horizon TVC: Discounting

With discounting, the TVC is easier to check.

Assume:

- the objective function is $e^{-\rho t} v [k(t), c(t)]$
- it only depends on $t$ through the discount factor
- $v$ and $g$ are weakly monotone

Then the TVC becomes

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) k(t) = 0 \tag{55}$$

where $\mu$ is the costate of the current value Hamiltonian.

This is exactly analogous to the discrete time version

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0 \tag{56}$$

# Example: renewable resource

# Example: Renewable resource

$$\begin{aligned} && \max \int_0^{\infty} e^{-\rho t} u(y(t))\, dt && (57)\\ && \textit{subject to} && (58)\\ && \dot{x}(t) = -y(t) && (59)\\ x(0) &=& 1 && (60)\\ x(t) &\geq& 0 && (61) \end{aligned}$$

# Example: Renewable resource

Current value Hamiltonian

Necessary FOCs

# Solution

Therefore:

$$\mu(t) = \mu(0)\, e^{\rho t} \tag{62}$$

$$y(t) = u'^{-1}\left[\mu(0)\, e^{\rho t}\right] \tag{63}$$

The optimal path has $\lim x(t) = 0$ or

$$\int_0^\infty y(t)\, dt = \int_0^\infty u'^{-1}\left[\mu(0)\, e^{\rho t}\right] dt = 1 \tag{64}$$

This solves for $\mu(0)$.

# Example: Renewable resource

TVC for infinite horizon case:

$$\lim e^{-\rho t}\mu(0)e^{\rho t}x(t)=0 \tag{65}$$

Equivalent to

$$\lim x(t)=0 \tag{66}$$

# Reading

- Acemoglu (2009), ch. 7. Proves the Theorems of Optimal Control.
- Barro and Sala-i Martin (1995), appendix.
- Leonard and Van Long (1992): A fairly comprehensive treatment. Contains many variations on boundary conditions.

# References I

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