Optimal Control

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Topics

Optimal control is a method for solving dynamic optimization problems in continuous time.

Example: Growth Model

A household chooses optimal consumption to

$$\max \int_0^T e^{-\rho t} u[c(t)]dt \tag{1}$$

subject to

$$\dot{k}(t) = rk(t) - c(t) \tag{2}$$

$$c(t) \in [0, \bar{c}] \tag{3}$$

$$k(0) = k_0, \text{given} \tag{4}$$

$$k(T) \geq 0 \tag{5}$$

Generic Optimal control problem

Choose functions of time $c(t)$ and $k(t)$ so as to

$$\max \int_0^T v[k(t), c(t), t] dt \tag{6}$$

Constraints:

1. Law of motion of the **state** variable $k(t)$:

$$\dot{k}(t) = g[k(t), c(t), t] \tag{7}$$

2. Feasible set for **control** variable $c(t)$:

$$c(t) \in Y(t) \tag{8}$$

3. Boundary conditions, such as:

$$k(0) = k_0, \text{given} \tag{9}$$

$$k(T) \geq k_T \tag{10}$$

Generic Optimal control problem

- c and k can be vectors.
- $Y(t)$ is a compact, nonempty set.
- T could be infinite.
    - Then the boundary conditions change
- Important: the state cannot jump; the control can.

A Recipe for Solving Optimal Control Problems

A Recipe

Step1. Write down the ***Hamiltonian***

$$H(t) = v(k,c,t) + \mu(t)\underbrace{g(k,c,t)}_{\dot{k}(t)} \tag{11}$$

μ is essentially a Lagrange multiplier (called a **co-state**).

Intuition:

- similar to the dynamic program: current utility + continuation value (but not quite)
- $v(k,c,t)$: current utility
- $\mu(t)$: the marginal value of increasing k for the future
- $g(k,c,t)$: captures how currenct actions affect future k

A Recipe

Step 2. Derive the **first order conditions** which are **necessary** for an optimum:

$$
\begin{aligned}
\partial H / \partial c &= 0 & (12) \\
\partial H / \partial k &= -\dot{\mu} & (13)
\end{aligned}
$$

Intuition below ...

A Recipe

Step 3. Impose the **transversality** condition:

- for finite horizon:

$$\mu(T) = 0 \tag{14}$$

- for infinite horizon:

$$\lim_{t \to \infty} H(t) = 0 \tag{15}$$

This depends on the terminal condition (see below).

A Recipe

Step 4. A **solution** is the a set of functions $[c(t), k(t), \mu(t)]$ which satisfy

- the FOCs
- the law of motion for the state
- the boundary / transversality conditions

Intuition $\partial H / \partial c = 0$

Maximize Hamiltonian w.r.to control.

Implies $v_c + \mu g_c = 0$

$v_c(k, c, t)$ picks up current utility of c

$\mu(t)$ is marginal value of additional "future" k.

$\mu(t) g_c(k, c, t)$ picks up change in continuation value (change in $\dot{k}$ times value of future k)

Intuition: $\partial H / \partial k = -\dot{\mu}$

Implies

$$v_k(k,c,t) + \mu g_k(k,c,t) = -\dot{\mu} \tag{16}$$

Think of this as $[\partial H / \partial k] / \mu = -\dot{\mu} / \mu$

- $\dot{\mu} / \mu$ is the growth rate of marginal utility
- $[\partial H / \partial k] / \mu$ is like a rate of return (marginal value of k now versus the future)
- if the rate of return is high, it is optimal to postpone consumption and let it grow
- then marginal utility declines over time

Example: Growth Model

$$\max \int_0^\infty e^{-\rho t} u(c(t))\,dt \tag{17}$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t) \tag{18}$$

$$k(0) \text{ given} \tag{19}$$

Growth Model: Hamiltonian

$$H(k,c,\mu) = e^{-\rho t}u(c(t)) + \mu(t)[f(k(t)) - c(t) - \delta k(t)] \tag{20}$$

Necessary conditions:

$$\begin{aligned} H_c &= e^{-\rho t}u'(c) - \mu = 0 \\ H_k &= \mu\left[f'(k) - \delta\right] = -\dot{\mu} \end{aligned}$$

Interpretation:

- μ is indeed the marginal value of capital (the same as the marginal value of consumption)
- $-g(\mu) = f'(k) - \delta$: when the rate of return is high, marginal utility falls over time

Substitute out the co-state

FOC imply:

$$g(\mu) = \delta - f'(k) \tag{21}$$

$$= \frac{d \ln \left(e^{-\rho t} u'(c_t)\right)}{dt} \tag{22}$$

$$= \frac{d}{dt}\left[-\rho t + \ln u'(c_t)\right] \tag{23}$$

$$= -\rho + \frac{u''(c_t)\,\dot{c}_t}{u'(c_t)} \tag{24}$$

Or:

$$\dot{c} = -\left(f'(k) - \delta - \rho\right)\frac{u'(c)}{u''(c)}$$

Solution: c_t, k_t that solve Euler equation and resource constraint, plus boundary conditions.

Details

First order conditions are necessary, not sufficient.

They are necessary only if we **assume** that

1. a continuous, interior solution exists;
2. the objective function v and the constraint function g are continuously differentiable.

Acemoglu (2009), ch. 7, offers some insight into why the FOCs are necessary.

Details

If there are multiple states and controls, simply write down one FOC for each separately:

$$\begin{aligned} \delta H/\delta c_i &= 0 \\ \partial H/\partial k_j &= -\dot{\mu}_j \end{aligned}$$

There is a large variety of cases depending on the length of the horizon (finite or infinite) and the kinds of boundary conditions.

- Each has its transversality condition (see Leonard and Van Long 1992).

Next steps

Typical useful next things to do:

1. Eliminate μ from the system. Obtain two differential equations in (c, k).
2. Find the steady state by imposing $\dot{c} = \dot{k} = 0$.

Sufficient conditions

First-order conditions are sufficient, if the programming problem is **concave**.

This can be checked in various ways.

Sufficient conditions I

The objective function and the constraints are concave functions of the controls and the states.

- The co-state must be positive.

This condition is easy to check, but very stringent.

In the growth model:

- $u(c)$ is concave in c (and, trivially, k)
- $f(k) - \delta k - c$ is concave in c and k
- $\mu = u'(c) > 0$

Sufficient Conditions II

(Mangasarian) First-order conditions are sufficient, if
the Hamiltonian is concave in controls and states,
where the co-state is evaluated at the optimal level (and held fixed).

This, too is very stringent.

In the growth model

$$\partial H/\partial c = u'(c) - \mu$$
$$\partial H/\partial k = \mu\,[f'(k) - \delta]$$
$$\partial^2 H/\partial c^2 = u''(c) < 0$$
$$\partial^2 H/\partial k^2 = \mu f''(k) < 0$$
$$\partial^2 H/\partial c \partial k = 0$$

Therefore: weak joint concavity (because we know that $\mu > 0$)

Sufficient Conditions III

Arrow and Kurz (1970)

- First-order conditions are sufficient, if the *maximized* Hamiltonian is concave in the states.
- If the maximized Hamiltonian is strictly concave in the states, the optimal path is unique.

Maximized Hamiltonian:

Substitute controls out, so that the Hamiltonian is only a function of the states.

This is less stringent and by far the most useful set of sufficient conditions.

In the growth model

Optimal consumption obeys $u'(c) = \mu$ or $c = u'^{-1}(\mu)$

Maximized Hamiltonian:

$$\hat{H} = u\left(u'^{-1}(\mu)\right) + \mu\left[f(k) - \delta k - u'^{-1}(\mu)\right] \tag{25}$$

We have $\partial \hat{H}/\partial k > 0$ and $\partial^2 \hat{H}/\partial k^2 = \mu f''(k) < 0$.

$\hat{H}$ is strictly concave in k.

Necessary conditions yield a unique optimal path.

Discounting: Current value Hamiltonian

Problems with discounting:

- Current utility depends on time only through an exponential discounting term $e^{-\rho t}$.

The generic discounted problem is

$$\max \int_0^T e^{-\rho t} v[k(t), c(t)] dt \tag{26}$$

subject to the same constraints as above.

Applying the Recipe

$$H(t) = e^{\rho t} v(k,c) + \hat{\mu} g(k,c) \tag{27}$$

$$\frac{\partial H}{\partial c_t} = 0 \implies e^{-\rho t} v_c(k_{t,} c_t) = -\hat{\mu}_t g_c(k_t, c_t) \tag{28}$$

$$\frac{\partial H}{\partial k_t} = e^{-\rho t} v_k(k_t, c_t) + \hat{\mu}_t g_k(k_t, c_t) = -\dot{\hat{\mu}}_t \tag{29}$$

Applying the Recipe

Let

$$\mu_t = e^{\rho t}\hat{\mu}_t \tag{30}$$

and multiply through by $e^{\rho t}$:

$$v_c(t) = -\mu_t g_k(t)$$

This is the standard FOC, but with μ instead of $\hat{\mu}$.

Applying the Recipe

$$v_k(t) + e^{\rho t}\hat{\mu}_t g_k(t) = -e^{\rho t}\dot{\hat{\mu}}_t \quad (31)$$

Substitute out $\dot{\hat{\mu}}_t$ using

$$\dot{\mu}_t = \frac{de^{\rho t}\hat{\mu}_t}{dt} = \rho\mu_t + e^{\rho t}\dot{\hat{\mu}}_t$$

we have

$$v_k(t) + \mu_t g_k(t) = -\dot{\mu}_t + \rho\mu_t$$

This is the standard condition with an additional $\rho\mu$ term.

Shortcut

We now have a shortcut for discounted problems.

Hamiltonian (drop the discounting term):

$$H = v(k,c) + \mu g(k,c) \tag{32}$$

FOCs:

$$\partial H / \partial c = 0 \tag{33}$$

$$\partial H / \partial k = \underbrace{\mu(t)\rho}_{\text{added}} - \dot{\mu}(t) \tag{34}$$

and the TVC

$$\lim_{T \to \infty} e^{-\rho T} \mu(T) k(T) = 0 \tag{35}$$

Equality constraints

Equality constraints of the form

$$h[c(t), k(t), t] = 0 \tag{36}$$

are simply added to the Hamiltonian as in a Lagrangian problem:

$$H(t) = v(k, c, t) + \mu(t) g(k, c, t) + \lambda(t) h(k, c, t) \tag{37}$$

FOCs are unchanged:

$$\begin{aligned} \partial H / \partial c &= 0 \\ \partial H / \partial k &= -\dot{\mu} \end{aligned}$$

For inequality constraints:

$$h(c, k, t) \geq 0; \; \lambda h = 0 \tag{38}$$

Transversality Conditions

Finite horizon: Scrap value problems

The horizon is T.

The objective function assigns a scrap value to the terminal state variable: $e^{-\rho T}\phi(k(T))$:

$$\max \int_0^T e^{-\rho t} v[k(t), c(t), t] dt + e^{-\rho T}\phi(k(T)) \tag{39}$$

Hamiltonian and FOCs: unchanged.

The TVC is

$$\mu(T) = \phi'(k(T)) \tag{40}$$

Intuition: μ is the marginal value of the state k.

Scrap value examples

1. Household with bequest motive

$$U = \int_0^T e^{\rho t} u(c(t)) + V(k_T) \tag{41}$$

with $\dot{k} = w + rk - c$.

2. Maximizing the present value of earnings

$$Y = \int_0^T e^{-rt} w h(t) [1 - l(t)] \tag{42}$$

subject to $\dot{h}(t) = A h(t)^{\alpha} l(t)^{\beta} - \delta h(t)$
Scrap value is 0. TVC: $\mu(T) = 0$.

Infinite horizon TVC

The finite horizon TVC with the boundary condition $k(T) \geq k_T$ is $\mu(T) = 0$.

- Intuition: capital has no value at the end of time.

But the infinite horizon boundary condition is NOT $\lim_{t \to \infty} \mu(t) = 0$.

The next example illustrates why.

Infinite horizon TVC: Example

$$
\begin{aligned}
& \max \int_0^{\infty} \left[\ln(c(t)) - \ln(c^*)\right] dt \\
& \textit{subject to} \\
\dot{k}(t) &= k(t)^{\alpha} - c(t) - \delta k(t) \\
k(0) &= 1 \\
\lim_{t\to\infty} k(t) &\geq 0
\end{aligned}
$$

c^* is the max steady state (golden rule) consumption.

No discounting - subtracting c^* makes utility finite.

Infinite horizon TVC

Hamiltonian

$$H(k,c,\lambda) = \ln c - \ln c^* + \lambda \left[k^\alpha - c - \delta k\right] \tag{43}$$

Necessary FOCs

$$\begin{aligned} H_c &= 1/c - \lambda = 0 && (44) \\ H_k &= \lambda \left[\alpha k^{\alpha-1} - \delta\right] = -\dot{\lambda} && (45) \end{aligned}$$

Infinite horizon TVC

We show: $\lim_{t\to\infty} c(t) = c^*$ [why?]

Limiting steady state solves

$$\begin{aligned} \dot{\lambda}/\lambda &= \alpha k^{\alpha-1} - \delta = 0 \\ \dot{k} &= k^{\alpha} - 1/\lambda - \delta k = 0 \end{aligned}$$

Solution is the golden rule:

$$k^* = (\alpha/\delta)^{1/(1-\alpha)} \tag{46}$$

Verify that this max's steady state consumption.

Infinite horizon TVC

Implications for the TVC...

$\lambda(t) = 1/c(t)$ implies $\lim_{t\to\infty} \lambda(t) = 1/c^*$.

Therefore, neither $\lambda(t)$ nor $\lambda(t)k(t)$ converge to 0.

The correct TVC:

$$\lim_{t\to\infty} H(t) = 0 \tag{47}$$

The only reason why the standard TVC does not work: there is **no discounting** in the example.

Infinite horizon TVC: Discounting

With discounting, the TVC is easier to check.

Assume:

- the objective function is $e^{-\rho t}v[k(t), c(t)]$
- it only depends on t through the discount factor
- v and g are weakly monotone

Then the TVC becomes

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)k(t) = 0 \tag{48}$$

where μ is the costate of the current value Hamiltonian.

This is exactly analogous to the discrete time version

$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0 \tag{49}$$

Example: renewable resource

Example: Renewable resource

$$\max \int_0^\infty e^{-\rho t} u(y(t))\, dt \tag{50}$$

$$\textit{subject to} \tag{51}$$

$$\dot{x}(t) = -y(t) \tag{52}$$

$$x(0) = 1 \tag{53}$$

$$x(t) \geq 0 \tag{54}$$

Example: Renewable resource

Current value Hamiltonian

Necessary FOCs

Example: Renewable resource

FOC

Therefore:

$$\mu(t) = \mu(0)\, e^{\rho t} \tag{55}$$

$$y(t) = u'^{-1}\left[\mu(0)\, e^{\rho t}\right] \tag{56}$$

Solution

The optimal path has $\lim x(t) = 0$ or

$$\int_0^\infty y(t)\,dt = \int_0^\infty u'^{-1}\left[\mu(0)\,e^{\rho t}\right]dt = 1 \tag{57}$$

This solves for $\mu(0)$.

Example: Renewable resource

TVC for infinite horizon case:

$$\lim e^{-\rho t}\mu(0)e^{\rho t}x(t)=0 \tag{58}$$

Equivalent to

$$\lim x(t)=0 \tag{59}$$

Reading

- Acemoglu (2009), ch. 7. Proves the Theorems of Optimal Control.
- Barro and Martin (1995), appendix.
- Leonard and Van Long (1992): A fairly comprehensive treatment. Contains many variations on boundary conditions.

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