1 Continuous Time CIA Model. Cash and Credit Goods.

Demographics: A single representative household who lives forever.

Endowments: At the beginning of time, households hold k_0 units of capital and M_0 units of money. Preferences:

$$\int_0^\infty e^{-\rho t} u\left(c_t, d_t\right) \, dt$$

where c and g are two consumption goods.

Technology:

$$f\left(k\right) = \dot{k} + c + d \tag{1}$$

Government: The government costlessly produces money M and hands it to households as lumpsum transfers. The money growth rate is constant at $g(M) = \gamma$. The government budget constraint is $\dot{M} = p x = g(M) M$ where p is the price level and x is the real lump-sum transfer.

Markets: There are competitive markets for goods and money. Households operate the technology (there are no firms).

CIA constraint: c has to be bought with cash:

$$c_t \leq M_t/p_t$$

d may be bought with credit.

Denote real balances by $m_t = M_t/p_t$.

Questions

1. Write down the household's Hamiltonian. Which are his states and controls? Derive firstorder conditions for two cases: either the CIA constraint always binds or it never binds. Hint: the budget constraint is given by

$$\dot{k}_t + c_t + d_t + \dot{M}_t / p_t = f(k_t) + x_t$$

- 2. Define a competitive equilibrium.
- 3. Derive a set of equations that characterize the steady state. Show that the nominal interest rate equals zero, if the CIA constraint does not bind.
- 4. Determine the effects of a higher money growth rate on the steady state allocation. Assume that the utility function takes the form u(c,d) = U(c) + V(d), where U and V are strictly concave functions.

1.1 Answer: Continuous Time CIA Model. Cash and Credit Goods.

1. The state is a = m + k. The controls are dg, m. The Hamiltonian is

$$H = u(c,g) + \lambda \left[f(a-m) + x - c - d - \pi m \right] + \varphi \left[m - c \right]$$

where $\pi = \dot{p}/p$ is the inflation rate. First-order conditions are:

$$u_{c} = \lambda + \varphi$$
$$u_{d} = \lambda$$
$$\lambda \left(f' \left(a - m \right) + \pi \right) = \varphi$$
$$\dot{\lambda} = \lambda \left(\rho - f' \left(a - m \right) \right)$$

These can be simplified as follows.

$$g(\lambda) = \rho - f'(k) = g(u_d) \tag{2}$$

$$u_c = u_d [1 + f'(k) + \pi]$$
(3)

A solution to the household problem then consists of a set of functions (c_t, d_t, m_t, k_t) that solve the first-order conditions (2) and (3), the budget constraint, and the CIA constraint if it binds, or $u_c = u_g$ if it does not.

Note: Making k the control and substituting out m from the budget constraint and CIA constraint works just as well.

- 2. A competitive equilibrium is a set of functions $(c_t, d_t, m_t, k_t, p_t, M_t, x_t)$ that solve 4 household equations, m = M/p, $\dot{M} = p x = g(M) M$, and goods market clearing $f(k) = c + d + \dot{k}$.
- 3. The steady state consists of scalars (c, d, m, k, π, x) which are determined near-recursively as follows. A constant real money stock requires $\pi = g(M)$. The Euler equation determines the capital stock: $f'(k) = \rho$. The two consumption flows are determined by f(k) = c + d and $u_c/u_d 1 = \rho + \pi$.

If the CIA constraint does not bind, then the last equation is $u_c = u_d$, so that $\pi = -\rho$ and the nominal interest rate is zero. This can only happen, if the money growth rate is set to $g(M) = -\rho$. If the CIA constraint binds, then m = c.

4. If the CIA constraint does not bind, then it will bind once the money growth rate is increased. So I only consider the effects when the CIA constraint binds. Clearly, money growth does not affect k. The effect on consumption is determined by f(k) = c + d and U'(c)/V'(d) = $1 + \rho + g(M)$. It is easy to see that higher money growth increases d and reduces c. Since m = c, real balances decline as well. Standard substitution is the intuition.

2 Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever. Preferences:

$$\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t) dt \tag{4}$$

where c is consumption and m denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with k_0 units of capital and m_0 units of real money.

Technology:

$$f(k_t) - \delta k_t = c_t + \dot{k}_t + \phi(\dot{m}) \tag{5}$$

where $\phi(\dot{m})$ is the cost of adjusting household money holdings. $\phi'(0) = 0$ and $\phi''(\dot{m}_t) > 0$.

Money: nominal money grows at exogenous rate g(M). New money is handed to households as a lump-sum transfer: $\dot{M}_t = p_t x_t$.

Markets: money (numeraire), goods, capital rental (price q), labor (w).

Questions:

1. The household's budget constraint is given by

$$\dot{k}_t + \dot{m}_t = w + r_t k_t + x_t - c_t - \pi_t m_t - \phi(\dot{m}_t)$$
(6)

where $r = q - \delta$. State the Hamiltonian. If you cannot figure this out, assume $\phi(\dot{m}) = 0$ and proceed (for less than full credit). Hint: Make m and k separate state variables.

- 2. State the first-order conditions.
- 3. Define a competitive equilibrium.
- 4. Characterize the steady state to the extent possible. What is the effect of a permanent change in g(M)?
- 5. What is the optimal rate of inflation? Explain.

2.1 Answer: Money in the Utility Function

1. We have to invent a control $z = \dot{m}$. Then

$$H = u(c,m) + \lambda \left[w + rk + x - c - \pi m - z - g(z) \right] + \mu z$$
(7)

2. FOCs

$$u_c = \lambda \tag{8}$$

$$\lambda \left[\phi'(z) + 1 \right] = \mu \tag{9}$$

$$\lambda = (\rho - r)\lambda \tag{10}$$

$$\dot{\mu} = \rho \mu - u_m + \lambda \pi \tag{11}$$

3. CE: $\{c, k, m, z, \lambda, \mu; w, r, \pi\}$ that satisfy

- household: 4 focs, 2 constraints, boundary conditions
- firm: standard focs
- goods market: feasibility
- capital and labor markets: implicit
- money growth: $g(M) = g(m) + \pi$

4. The BGP is recursive: The Euler equation fixes $r = \rho$. The firm's foc fixes k. From k we have w, output, and $c = f(k) - \delta k$. Money is super-neutral.

Constant real money fixes $\pi = g(M)$. That only leaves the money demand equation $u_m = (\rho - g(M))u_c$ to determine m. Higher g(M) raises inflation and changes money demand, m. Real variables are not affected.

5. Optimal inflation rate: Friedman rule. Saturate the household with money.