Problem Set 5: Growth Model in Continuous Time

Econ720. Fall 2021. Prof. Lutz Hendricks

1 Relative Wealth Preferences

Consider the following version of the growth model in continuous time.

Notation: c: consumption, k: capital, $\bar{k}$: average capital in the economy.

Demographics: There is one representative household who lives forever.

Preferences:

$$\int_0^\infty e^{-\rho t} \left[U(c_t) + V\left(k_t/\bar{k}_t\right)\right] dt \tag{1}$$

Endowments: The household starts with k_0.

Technology:

$$\dot{k}_t = f(k_t) - c_t \tag{2}$$

Government budget constraint: The government taxes consumption at rate τ_c and lump-sum rebates the revenues R_t to the household.

$$R_t = \tau_c c_t \tag{3}$$

Markets: Goods (numeraire).

Household budget constraint:

$$\dot{k}_t = f(k_t) - (1+\tau_c) c_t + R_t \tag{4}$$

Assumptions: U, V, f are strictly increasing and strictly concave. $f'(0) = \infty$. $f'(\infty) = 0$.

Questions:

1. State the household's current value Hamiltonian and derive the first-order conditions. Do not yet substitute out the co-state. Define a solution to the household problem.
2. Define a competitive equilibrium. Make sure that the number of objects equals the number of equations.
3. Derive an equation that implicitly solves for the steady state capital stock.
4. Draw the phase diagram. Start with $\dot{k} = 0$ and discuss its shape.
5. Derive $\dot{c} = 0$ and discuss its slope / intercept. For which values of k does $\dot{c} = 0$ have a solution? Hint: It is easier to write down $\dot{\lambda} = 0$, where λ is the co-state. Then use the fact that $\dot{\lambda} > 0$ implies $\dot{c} < 0$.
6. Assume that $\dot{c} = 0$ is concave,
$$\partial^2 c/\partial k^2|_{\dot{c}=0} < 0 \tag{5}$$
and that it intersects $\dot{k} = 0$ twice. Discuss the stability properties of the two steady states.

1.1 Answer: Relative wealth preferences

1. This question is based on Wen-ya Chang, "Relative wealth, consumption taxation, and economic growth," *Journal of Economics* 2006, 88(2): 103-29.

$$H = U(c) + V\left(k/\bar{k}\right) + \lambda\left[f(k) - R - (1+\tau_c) c\right] \tag{6}$$

FOC:

$$U'(c) = \lambda(1+\tau_c) \tag{7}$$

$$V'\left(k/\bar{k}\right)/\bar{k} + \lambda f'(k) = \rho\lambda - \dot{\lambda} \tag{8}$$

Solution: c, k, λ that solve 2 FOCs, budget constraint, TVC: $\lim e^{-\rho t}\lambda_t k_t = 0$.

2. CE: $c, k, \bar{k}, \lambda, R$ that solve: household (3), government budget constraint, goods market clearing

$$f(k) = c + \dot{k} \tag{9}$$

and the identity $k = \bar{k}$.

3. Steady state: Set $\dot{\lambda} = 0$ and $f(k) = c$:

$$V'(1)/k = \frac{U'(f(k))}{1+\tau_c}\left[\rho - f'(k)\right] \tag{10}$$

Relative to the case of $V = 0$, steady state capital is higher.

4. Phase diagram. $\dot{k} = 0$ requires $c = f(k)$. Simply plot the production function. Higher c reduces $\dot{k}$.

5. $\dot{\lambda} = 0$ requires

$$\lambda = \frac{U'(c)}{1+\tau_c} = \frac{V'(1)}{k\left[\rho - f'(k)\right]} \tag{11}$$

Since $\lambda > 0$, this is only defined for sufficiently high k such that $\rho - f'(k) > 0$. The slope of $k\left[\rho - f'(k)\right]$ is $\left[\rho - f'(k)\right] - kf''(k) > 0$. Therefore, $\partial\lambda/\partial k|_{\dot{c}=0} < 0$ and $\partial c/\partial k|_{\dot{c}=0} > 0$. From

$$\dot{\lambda} = \lambda\left[\rho - f'(k)\right] - V'(1)/k \tag{12}$$

it follows that higher λ raises $\dot{\lambda}$. Thus, higher c raises $\dot{c}$.

6. Draw the phase diagram. It should look like figure 1. The lower steady state is saddle path stable. To show this, argue that any path not leading to the steady state would violate a boundary condition. The tricky part here: showing that the area where $\dot{c} < 0$ and $\dot{k} > 0$ cannot be reached. It would violate TVC. To see this write

$$\begin{aligned} g(\lambda) + g(k) &= \rho - \frac{V'(1)}{\lambda k} - f'(k) + \frac{f(k) - c}{k} \\ &= \rho - \frac{V'(1)/\lambda + c}{k} + \frac{f(k) - f'(k)k}{k} \end{aligned}$$

Note that for large enough k, $g(\lambda) + g(k) > \rho$. This happens because $V'(1) + c$ are falling and $f(k) - f'(k)k$ [which equals labor income] is rising over time.

The upper steady state is unstable. It has ever rising c and k. Can this exist? A path close to $\dot{\lambda} = 0$ satisfies the TVC, which implies

$$V'(1) = \lambda k\left[\rho - f'(k)\right]$$

or asymptotically $\lambda k \to V'(1)/\rho$. If parameters are such that $c < f(k)$ and $c > u'^{-1}\left(\frac{V'(1)(1+\tau_c)}{\rho k}\right)$ can be simultaneously satisfied as $k \to \infty$, then such an equilibrium exists. If not, then the phase diagram does not look like figure 1.

2 Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever.

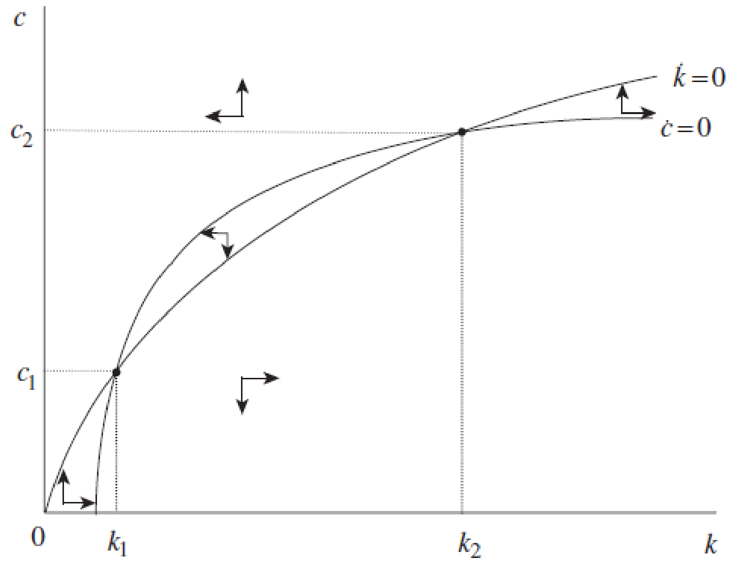
Preferences:

$$\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t)\, dt \tag{13}$$

where c is consumption and m denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with k_0 units of capital and m_0 units of real money.

Figure 1: Phase diagram



Technology:

$$f(k_t) - \delta k_t = c_t + \dot{k}_t \tag{14}$$

Money: nominal money grows at exogenous rate $g(M)$. New money is handed to households as a lump-sum transfer: $\dot{M}_t = p_t x_t$.

Markets: money (numeraire), goods, capital rental (price r), labor (w).

Questions:

1. The household's budget constraint is given by

$$\dot{k}_t + \dot{m}_t = w + r_t k_t + x_t - c_t - \pi_t m_t - \phi(\dot{m}_t) \tag{15}$$

   where $\phi(\dot{m}_t)$ is the cost of adjusting the money stock. $\phi'(0) = 0$ and $\phi''(\dot{m}_t) > 0$. State the Hamiltonian. If you cannot figure this out, assume $\phi(\dot{m}) = 0$ and proceed (for less than full credit).

2. State the first-order conditions.

3. Define a competitive equilibrium.

4. Characterize the steady state to the extent possible. What is the effect of a permanent change in $g(M)$?

5. What is the optimal rate of inflation? Explain.

2.1 Answer: Money in the Utility Function

1. We have to invent a control $z = \dot{m}$. Then

$$H = u(c, m) + \lambda \left[w + rk + x - c - \pi m - z - g(z)\right] + \mu z \tag{16}$$

2. FOCs

$$\begin{aligned} u_c &= \lambda && (17) \\ \lambda\left[\phi'(z) + 1\right] &= \mu && (18) \\ \dot{\lambda} &= (\rho - r)\lambda && (19) \\ \dot{\mu} &= \rho\mu - u_m + \lambda\pi && (20) \end{aligned}$$

3. CE: $\{c, k, m, z, \lambda, \mu; w, r, \pi\}$ that satisfy

    - household: 4 focs, 2 constraints, boundary conditions
    - firm: standard focs
    - goods market: feasibility
    - capital and labor markets: implicit
    - money growth: $g(M) = g(m) + \pi$

4. The BGP is recursive: The Euler equation fixes $r = \rho$. The firm's foc fixes k. From k we have w, output, and $c = f(k) - \delta k$. Money is super-neutral.

    Constant real money fixes $\pi = g(M)$. That only leaves the money demand equation $u_m = (\rho - g(M))u_c$ to determine m. Higher $g(M)$ raises inflation and changes money demand, m. Real variables are not affected.

5. Optimal inflation rate: Friedman rule. Saturate the household with money.