The Growth Model: Discrete Time Planner's Dynamic Program

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The Planner's Problem with DP

The Planner's Bellman equation is

$$V(k) = \max_{c} u(c) + \beta V(f(k) - c)$$

with state k and control c.

The FOC for c is

$$u'(c) = \beta \; V'(k')$$

Problem: we do not know V'.

The Planner's Problem with DP

Differentiate the Bellman equation to obtain the
envelope condition
(aka Benveniste-Scheinkman equation):

$$V'(k) = \beta V'(k') f'(k) + \frac{\partial c}{\partial k} \underbrace{\left[u'(c) - \beta V'(k')\right]}_{=0}$$

Key point:

in the envelope condition, we can always ignore that changing the state (k) affects the controls (c).

The Planner's Problem with DP

Combine the FOC and the envelope condition to sub out all terms involving V':

$$\beta V'\left(k'\right) = \beta\beta V'\left(k''\right) f'\left(k'\right) \tag{1}$$

$$u'\left(c\right) = \beta u'\left(c'\right) f'\left(k'\right) \tag{2}$$

We obtain the same Euler equation as from the Lagrangian approach (of course).

DP also tells us that the optimal c is a function only of k.

Therefore k' also depends only on k:

$$\begin{aligned} k' &= f(k) - \phi(k) \\ &= h(k) \end{aligned}$$

Capital as control variable

There are other ways of setting up the Bellman equation.

With capital as the control:

$$V(k) = \max_{k'} u(f(k) - k') + \beta V(k')$$

FOC:

$$u'(c) = \beta V'(k')$$

Envelope condition

$$V'(k) = u'(c) f'(k)$$

The general point: We cannot choose the state variables, but we can choose the control variables.

Characterizing the Planner's Solution

It is here where DP has serious advantages over the Lagrangian: one can use results from **functional analysis** to establish properties of the value function and the policy function.

In our example, it can be shown that the economy converges monotonically from any k_0 to the steady state [Sargent (2009), p. 25, fn. 2]:

Note the difference relative to the OLG economy where much stronger assumptions are needed for this result.

Nonstationary Dynamic Programming

What if time matters?

Case 1: Time matters because of a **time-varying state variable**.

- Example: $f(k_t, A_t)$ where $A_{t+1} = G(A_t)$.
- Solution: Add A_t as a state variable to the value function.

Case 2: **Finite horizon** problems.

- Example: the household lives until date T.
- Solution: Add t as a state variable to the value function.
- We have one value function per date (see below).

Additional Constraints

Constraints are treated as in any optimization problem.

Example:

$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to

- $k' = f(k) - c$
- $k' \geq 0$

Bellman equation:

$$V(k) = \max_{c,k'} u(c) + \beta V(k') + \lambda \left(f(k) - c - k'\right) + \mu k' \tag{3}$$

First-order conditions: Kuhn Tucker for k'.

Example: Non-separable Utility

Example: Non-separable Utility

Consider the following growth economy, modified to include **habit persistence** in consumption.

The social planner solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1})$$

subject to the feasibility constraints

$$k_{t+1} + c_t = f(k_t) \tag{4}$$

f satisfies Inada conditions.

Compute and interpret the first-order necessary conditions for the planner's problem.

Sequential Solution

This problem does not fit the DP approach without some modification.

We first solve it using a Lagrangian:

$$\Gamma = \sum_{t=1}^{\infty} \beta^t u(f(k_t) - k_{t+1}, f(k_{t-1}) - k_t) \tag{5}$$

First order conditions:

$$\beta^t u_1(t, t-1) f'(k_t) - \beta^{t-1} u_1(t-1, t-2)$$
$$+ \beta^{t+1} u_2(t+1, t) f'(k_t) - \beta^t u_2(t, t-1) = 0$$

Sequential Solution

Define the total marginal utility of consumption as

$$U'(c_{t-1}) = u_1(t-1, t-2) + \beta u_2(t, t-1)$$

The Euler Equation then becomes:

$$U'(c_{t-1}) = \beta U'(c_t) f'(k_t) \tag{6}$$

Interpretation

$$U'(c_{t-1}) = \beta U'(c_t) f'(k_t) \tag{7}$$

- Give up one unit of c_{t-1}. This costs $U'(c_{t-1})$.
- We can increase x_{t-1} by 1 and raise k_t by 1.
- We eat the results next period at marginal utility $U'(c_t)$.
- We can eat
  - the additional output $f'(k_t)$;
  - the undepreciated capital $1-\delta$; (zero, in this case)

Sequential Solution

A **solution** of the hh problem is:

Sequences $\{c_t, k_t\}$ that satisfy

1. the EE
2. the flow budget constraint.
3. The boundary conditions k_1 given and a TVC:

$$\lim_{t \to \infty} \beta^t U'(c_t) k_t = 0$$

DP Solution

For DP to work, it must be possible to write the problem as

$$V(s) = \max_{c} \; u(s,c) + \beta \; V(s')$$

subject to $s' = g(s,c)$

where s is the state and c is the control.

The current problem does not fit that pattern:

$$V(k) = \max \; u(c, c_{-1}) + \beta \; V(k')$$

subject to the law of motion

$$k' \;\; = \;\; f(k) - c$$

Nonseparable utility is the problem.

Adding a State Variable

The solution is to define an additional state variable

$$z = c_{-1}$$

or

$$z' = c = f(k) - k' \tag{8}$$

Then the Bellman equation is

$$V(k,z) = \max_{k'} u\left(f(k) - k', z\right) + \beta V\left(k', f(k) - k'\right)$$

Note that this looks “wrong” b/c z appears only once on the RHS, but everything is fine...

FOC

$$u_1(c,z) = \beta V_k(k',z') - \beta V_z(k',z')$$

Adding a state variable

The envelope conditions are

$$\begin{aligned} V_z &= u_2(c,z) \\ V_k &= u_1(c,z)f'(k) + \beta V_z(.')f'(k) \end{aligned}$$

Now define

$$U'(c) = u_1(c,z) + \beta u_2(c',z')$$

Then substitute out the V_z terms:

$$\begin{aligned} U'(c) &= \beta V_k(.') \\ V_k &= U'(c)f'(k) \end{aligned}$$

Substitute out the V_k terms and we get the same EE as with the Lagrangian.

The key point

If lagged variables occur in the problem, simply define new variables for date t: $z_t = c_{t-1}$.

Guess and Verify

Guess and Verify

In very special cases it is possible to solve for the value function in closed form.

A common case is

- log utility, $u(c) = \ln(c)$, and
- Cobb-Douglas technology with full depreciation: $f(k) = Ak^{\theta}$.

Then we can use the "guess and verify" method.

Guess and Verify

The general approach is:

1. Guess a functional form for V. Stick this into the right-hand-side of the Bellman equation.
2. Solve the max problem given the guess for V. The result is on the left hand side a new value function, V^1.
3. If $V = V^1$ the guess was correct.

This nicely illustrates what defines a solution to the Bellman equation.

Guess and Verify: Example

Consider the growth model with log utility and Cobb-Douglas production / full depreciation.

The planner solves:

$$\max \sum_{t=0}^{\infty} \beta^t \ \ln(c_t)$$

$$s.t. \quad k_{t+1} = A\, k_t^{\theta} - c_t$$

Guess

Guess

$$V(k) = E + F \ln(k)$$

This is inspired by the hope that V should inherit the form of u.
Having capital stock k amounts to having output Ak^{θ}, which would suggest

$$\begin{aligned} V(k) &\cong \ln(Ak^{\theta}) \\ &= \ln(A) + \theta \ln(k) \end{aligned}$$

Note that the guess for V contains some unknown constants (E, F) which we determine as we go along.

First-order Conditions

FOC:

$$u'(c) = \beta V'(k')$$

or

$$1/c = \beta F/k' \tag{9}$$

Envelope condition

$$V'(k) = u'(c)f'(k)$$

or

$$F/k = f'(k)/c \tag{10}$$

Policy Function

We can use the FOC to obtain the policy function in terms of the unknown parameters.

$$Fc = k'/\beta = f'(k)k \tag{11}$$

Note that

$$f'(k)k = \theta f(k) \tag{12}$$

Here, we are lucky and the F drops out

$$k' = h(k) = \beta\theta f(k) \tag{13}$$

$$c = (1-\beta\theta)f(k) \tag{14}$$

Result (as expected): the saving rate is constant.

Recover F

Now we need to recover E and F (and make sure they are indeed constants)

We know:

$$Fc = k'/\beta \tag{15}$$

$$c/k' = F/\beta \tag{16}$$

From the policy rules:

$$c/k' = (1-\beta\theta)/\theta \tag{17}$$

Therefore

$$F = \frac{\theta}{1-\theta\beta} \tag{18}$$

Recover E

Substitute everything we know into the Bellman equation:

$$E + F\ln(k) = \ln((1-\beta\theta)f(k)) + \beta\{E + F\ln(\beta\theta f(k))\} \qquad (19)$$

Note that $\ln(f(k)) = \ln(A) + \theta\ln(k)$.
Collect all the constant terms to solve for E

$$E = \ln(1-\theta\beta) + \ln(A) + \beta E + \beta F\ln(\theta\beta) + \beta FA \qquad (20)$$

Summary: Guess and Verify

1. Guess a value function (including unknown parameters).
2. Write first-order and Envelope conditions using the guess.
3. Solve for policy function.
4. Substitute policy function into Bellman equation to recover unknown parameters (and check the guess).

Applications

Examples where guess + verify is used:

Huggett et al. (2006), Huggett et al. (2011), Manuelli and Seshadri (2014)

(all models of human capital accumulation)

DP vs Lagrangian

What does DP buy us compared with a Lagrangian?

- With **uncertainty**, DP tends to be more convenient than a Lagrangian.
- Results from functional analysis can often be used to find **properties** of the optimal policy function such as monotonicity, continuity, and existence.
- DP can have **computational** advantages. There are methods for numerically approximating policy functions.

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Ljungqvist and Sargent (2004), ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- Stokey et al. (1989), ch. 1 is a nice introduction.

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