Review Questions: Asset Pricing

Econ720. Fall 2021. Prof. Lutz Hendricks

1 Deterministic Fruit Tree Prices

Consider the following version of a "fruit tree" economy a la Lucas. There is a unit measure of identical, infinitely lived households. Each receives an endowment of e_t in each period. In addition, each household owns one tree at the beginning of time (date 1) which yields d_t units of the consumption good in each period. Goods cannot be stored, but trees live forever. Trees can be bought or sold at a price of p_t, which is of course endogenous. The number of trees in the economy cannot be altered; it is always 1 per household.

Hence, the household solves

$$\max \sum\nolimits_{t=1}^{\infty} \beta^t u(c_t)$$

subject to

$$p_t k_{t+1} = k_t(d_t + p_t) + e_t - c_t$$

with $k_1 = 1$. In the budget constraint, the household receives as income the endowment e_t and capital income proportional to the number of trees owned (k_t). This is spent on consumption ct and the purchase of new trees (k_{t+1}).

(a) State the household problem as a Dynamic Program.

(b) State the conditions that define a solution to the household problem.

(c) Define a competitive equilibrium.

(d) Assume that e_t and d_t are constant over time. Derive the steady state price of a tree (p) and the steady state rate of return of holding a tree.

1.1 Answer: Deterministic Fruit Tree Prices

(a) Bellman equation

$$V(k) = \max u\left(e + k(d+p) - pk'\right) + \beta V(k')$$

(b) A solution consists of sequences (c_t, k_t) that satisfy the budget constraint and the Euler equation

$$u'(c) = \beta u'(c')(d' + p')/p$$

(c) A CE consists of sequences (c_t, k_t, p_t) that satisfy the 2 household optimality conditions and the market clearing conditions $k_t = 1$ and $c_t = e_t + d_t$.

(d) The EE implies that $p = d\beta/(1-\beta)$.

2 Deterministic Asset Prices

Consider the following deterministic Lucas fruit tree economy. There is a single representative agent who values consumption streams according to $\sum_{t=0}^{\infty} \beta^t u(c_t)$. There are (measure) $K_i > 0$ fruit trees of type $i \in \{1, 2\}$ which yield exogenous dividend streams of $d_{it} > 0$ units of the good. Trees are traded at (endogenous) prices p_{it}. The household also receives an endowment stream e_t.

(a) Solve the household problem using Dynamic Programming. Note that the household's budget constraint is

$$e_t + \sum_{i=1}^{2} (p_{it} + d_{it})\, k_{it} = c_t + \sum_{i=1}^{2} p_{it}\, k_{i,t+1}$$

where k_{it} denotes the amount of type i trees held by the household in period t.

(b) Define a competitive equilibrium.

(c) Solve in closed form for the prices of all trees in terms of (future) endowments and dividends. *Hint:* Note that the marginal rate of substitution terms $\alpha_{t+j} = u'(c_{t+j})/u'(c_t)$ are essentially exogenous because agents eat essentially exogenous amounts in each period in equilibrium.

(d) Suppose that endowments and dividends cycle. The endowments cycle between $e_t = e^{odd}$ in odd periods and $e_t = e^{even}$ in even periods. Assume $e^{odd} < e^{even}$. Tree 1 pays $d_{1t} = d$ in odd periods and nothing in even periods. Tree 2 pays $d_{2t} = d$ in even periods and nothing in odd periods. Calculate the asset price ratio, p_{1t}/p_{2t}, for odd and even t. Explain what you find. If you can't solve this, explain what you would expect to find.

2.1 Answer: Deterministic Asset Prices

(a) The Bellman equation is

$$V(k_1, k_2) = \max\, u\left(e + \sum (p_i + d_i)\, k_i - \sum p_i\, k_i'\right) + \beta\, V(k_1', k_2')$$

The Euler equation is standard:

$$u'(c) = \beta\, R_i'\, u'(c')$$

where $R_i' = (p_i' + d_i')/p_i$ is the rate of return of a type i tree. A solution consists of sequences c_t and $a_t = \sum p_{it}\, k_{it}$ which satisfy the Euler equation and the flow budget constraint (and a TVC). Note that the portfolio composition is indeterminate.

(b) A competitive equilibrium is a set of sequences (c_t, k_{it}, p_{it}) that satisfy: 2 household conditions; $k_{it} = K_{it}$; $c_t = e_t + \sum d_{it}\, k_{it}$. Finally, for asset markets to clear, all trees must yield the same rate of return: $R_{it} = R_t$.

(c) This is standard iterating over a difference equation stuff:

$$\begin{aligned} p_{it} &= \beta\, \alpha_{t+1}\, (d_{it+1} + p_{it+1}) \\ &= \sum_{j=1}^{\infty} \beta^j\, \alpha_{t+j}\, d_{it+j} \end{aligned} \qquad (1)$$

where $\alpha_{t+j} = u'(c_{t+j})/u'(c_t)$ is the marginal rate of substitution (which is essentially exogenous).

(d) Note that α_{t+j} now takes on only 2 values. If t is even, then $c_t = c^{even} = e^{even} + d\, K_2$.
If t is odd, then $c_t = c^{odd} = e^{odd} + d\, K_1$. Therefore, if t is odd, then $\alpha_{t-1+2j} = \alpha^{odd} = u'(c^{even})/u'(c^{odd})$ and $\alpha_{t+2j} = 1$. But if t is even, then $\alpha_{t-1+2j} = \alpha^{even}$ and $\alpha_{t+2j} = 1$. In words: The MRS oscillates between two values. This is helpful for pricing the assets because each asset pays either in periods with $MRS = \alpha^{even}$ or with $MRS = \alpha^{odd}$.

Now consider an **even period** t. Asset 1 pays in odd periods $(t+1, t+3, ...)$. Hence, from (1), its price is given by

$$p_1^{even} = \alpha^{even}\, d\, \left[\beta + \beta^3 + ...\right] = \beta \sum_{j=0}^{\infty} \beta^{2j}\, \alpha^{even}\, d = \frac{\beta\, \alpha^{even}\, d}{1 - \beta^2}$$

Here I used the fact that

$$\sum_{j=0}^{\infty} \beta^{2j} = \sum_{j=0}^{\infty} (\beta^2)^j = \frac{1}{1 - \beta^2}$$

Similarly, tree 2 pays in even periods $(t+2, t+4, ...)$ and its price in an even period t must be

$$p_2^{even} = d\, \beta^2\, \left[1 + \beta^2 + ...\right] = \beta^2 \sum_{j=0}^{\infty} \beta^{2j}\, d = \frac{\beta^2\, d}{1 - \beta^2}$$

Hence, the price ratio in even periods is given by

$$PR^{even} = \frac{p_1^{even}}{p_2^{even}} = \frac{\alpha^{even}}{\beta}$$

If t is odd, the calculation is similar and yields

$$
\begin{aligned}
p_1^{odd} &= \frac{\beta^2\, d}{1-\beta^2} \\
p_2^{odd} &= \frac{\beta\, \alpha^{odd}\, d}{1-\beta^2} \\
PR^{odd} &= \frac{\beta}{\alpha^{odd}}
\end{aligned}
$$

Assuming that $c^{odd} < c^{even}$, it follows that $\alpha^{odd} < 1$ and $\alpha^{even} > 1$. Therefore, $PR^{even} > 1$ and $PR^{odd} < 1$. The intuition is simple. Tree 1 yields fruit when consumption is low and marginal utility is high. Hence, it tends to be more valuable than tree 2. In even periods, tree 1 has the additional advantage of paying dividends one period earlier that tree 2 (in $t+1$ instead of $t+2$). Hence, $PR^{even} > 1$. However, in odd periods, tree 1 pays dividends one period later than tree 2 and PR^{odd} may be less than 1.