The Growth Model: Discrete Time

Prof. Lutz Hendricks

Econ720

September 10, 2021

The standard growth model

- The neoclassical growth model, aka the standard growth model, is the most important model in macro.
- It underlies entire branches of the literature (parts of growth theory and business cycle theory, for example).
- Here, we study this model in discrete time.
- The **main issues** of this section are:
  - Tools: Dynamic programming
  - The neoclassical growth model

Model structure

There are many versions of the growth model. This is a basic version.

1. Households are identical and live forever.
2. Firms produce a single good using capital and labor.
3. All agents are price takers.
4. All prices are perfectly flexible. All markets clear at all times.

Infinite horizons

- So far we have assumed that agents are finitely lived.
- Analytically more convenient: infinite lifetimes.
- How to justify this?
  - Reduced form of an OLG model with **altruism**.
  - Stochastic deaths (perpetual youth models).
  - But really: convenience + show it does not matter.

Demographics

There is a continuum of households (uncountably infinite number).

All households are identical.

- This is stronger than needed (see notes on aggregation later on).

We can think of a single, price-taking household.

The measure of households is 1.

Therefore, per capita and aggregate variables are the same.

Exercise: Redo everything when the number of households is $N_t = (1+n)^t$.

Preferences

The household values discounted utility from consumption:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1 \tag{1}$$

Utility is time separable (for tractability).

Discounting is exponential (to avoid time consistency problems).

Time consistency means:

- If $\{c_t\}_{t=0}^{\infty}$ solves the problem with start date 0, then $\{c_t\}_{t=\tau}^{\infty}$ solves the problem with start date τ.
- The household does not want to change past plans.

Endowments

The household has

- k_0 units of the good at $t = 0$
- 1 unit of time in each period

Technology

- Resource constraint:

$$k_{t+1} = f(k_t) - c_t \tag{2}$$

- We assume Inada conditions for f.
- Capital cannot be negative: $k_t \geq 0$.

Markets

Goods: numeraire.

Labor: w_t

Capital rental: q_t

All markets are competitive.

Planning Problem

The planner maximizes discounted utility of the representative household

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

Constraints:

$$\begin{aligned} k_{t+1} &= f(k_t) - c_t \\ k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

Lagrangian

$$
\begin{aligned}
\Gamma &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&\quad + \sum_{t=0}^{\infty} \lambda_t [f(k_t) - c_t - k_{t+1}]
\end{aligned}
$$

FOCs for an interior solution:

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t \\
\lambda_{t+1} f'(k_{t+1}) &= \lambda_t
\end{aligned}
$$

Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}) \tag{3}$$

This is exactly the same Euler equation we saw many times before.

The Euler equation implicitly defines a law of motion for the capital stock:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \tag{4}$$

This is a second order difference equation.

- but later we will see that it isn't...

Planner: Solution

A solution is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

These satisfy the **necessary** conditions:

1. Euler equation
2. Resource constraint
3. k_0 given

We have two difference equations, but only one **boundary condition.**

Uniqueness requires an additional restriction:

Transversality:

$$\lim_{t\to\infty} \beta^t \, u'(c_t) \; k_{t+1} = 0 \tag{5}$$

Digression: Transversality Conditions

Digression: Transversality Conditions

Consider the following example:

$$\begin{aligned} & \max \sum_{t=0}^{T} \beta^t u(c_t) \\ s.t. \;\; k_{t+1} &= e_t + (1+r_t)k_t - c_t \end{aligned}$$

Note the finite horizon: $T < \infty$.

As stated, this problem does not have a solution (why not?).

Digression: Transversality Conditions

Let's proceed to solve the problem as stated.

Lagrangian

$$\begin{aligned}\Gamma &= \sum_{t=0}^{T} \beta^t u(c_t) \\ &\quad + \sum_{t=0}^{T} \lambda_t \{e_t + (1+r_t) k_t - c_t - k_{t+1}\}\end{aligned}$$

FOCs (necessary):

$$u'(c_t) = \beta\ u'(c_{t+1})\ (1+r_{t+1})$$

Solution

Sequences $\{c_t, k_{t+1}\}$ that satisfy:

- Euler equation
- budget constraint
- k_0 given

Problems

Problem 1:

- We allowed the household to choose $c_t \to \infty$ and $k_{t+1} \to -\infty$.
- The household problem has no solution.

Problem 2:

- We have 2 difference equations, but only one boundary condition.
- The solution is not uniquely determined by those.

We need one more boundary condition to ensure that utility is finite.

Where to Find a Boundary Condition?

The economics of the problem must suggest the right condition.

It needs to be imposed as part of the original problem with some economic justification.

A natural candidate in this example: $k_{T+1} = 0$.

- The household cannot die in debt.

Infinite horizon case

What if $T \to \infty$?

We could impose $\lim_{T\to\infty} k_{T+1} = 0$, but it does not make economic sense.

- This would prevent the household from perpetually growing its capital stock.

We need to find a weaker condition that makes utility finite.

Infinite horizon case

One solution:

Write the present value budget constraint as

$$\sum_{t=0}^{T} \frac{c_t}{R_t} = \sum_{t=0}^{T} \frac{e_t}{R_t} + k_0 - \frac{k_{T+1}}{R_{T+1}}$$

where $R_t = (1+r_1) \times ... \times (1+r_t)$ is a cumulative discount factor.

- **Exercise**: show that this is the present value budget constraint.

Require that $\lim_{T\to\infty} k_{T+1}/R_{T+1} = 0$.

- That ensures finite consumption and utility

Note: the solution may still not be unique. But at least there is one.

Infinite horizon case

An equivalent solution:

Impose

$$\lim_{T \to \infty} \beta^T \; u'(c_T) \; k_{T+1} = 0$$

This is the same because, by the Euler equation:

$$\beta^T u'(c_T) R_T = u'(c_0)$$

The general point

In dynamic optimization problems, the flow budget constraint is usually not enough to ensure finite utility.

With finite T, the boundary condition is usually obvious (e.g.: cannot die in debt).

With infinite T, the boundary condition usually restricts debt at $t \to \infty$.

In both cases, the boundary condition is part of the economic environment.

Final note

For the planner's problem, we have $k_t \geq 0$.

- The TVC is satisfied automatically.

But without the TVC, we could have a path that has

- $k_t \to k_{ss}$ and $c_t \to 0$
- and that satisfies Euler and resource constraint

That path would not be optimal, even though it satisfies all necessary conditions.

Adding the TVC rules out such paths (but that is a matter of sufficiency).

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Stokey et al. (1989), ch. 1 is a nice introduction.
- Blanchard and Fischer (1989) is a good introduction to the standard growth model.
- Krusell (2014) ch. 2 discusses why the assumptions made in the growth model are popular.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Krusell, P. (2014): “Real Macroeconomic Theory,” Unpublished.

Stokey, N., R. Lucas, and E. C. Prescott (1989): “Recursive Methods in Economic Dynamics,” .