Problem Set 3: Infinite Horizon Model in Discrete Time

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1 Wealth in the utility function

Consider the following modification of the standard growth model where the households derives utility from holding wealth.

Demographics: There is a representative household of unit mass who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$ where c_t is consumption and k_{t-1} is last period's capital (wealth). The utility function is strictly concave and increasing in both arguments.

Endowments: At $t = 0$ the household is endowed with capital K_0. In each period the household works 1 unit of time.

Technologies:

$$K_{t+1} = A\,F(K_t, L_t) + (1-\delta)\,K_t - c_t \tag{1}$$

The production function has constant returns to scale.

Markets: Production takes place in a representative firm which rents capital and labor from households. There are competitive markets for goods (price 1), capital rental (r_t), and labor rental (w_t).

1. State the household's dynamic program.
2. Derive and explain the conditions that characterize a solution to the household problem (in sequence language).
3. Define a competitive equilibrium.
4. Derive a single equation that determines the steady state capital stock.
5. Is the steady state unique? Explain the intuition why the steady state is or is not unique.

1.1 Answer: Wealth in the utility function

(a) Bellman equation

$$V(k,z) = \max\; u(c,z) + \beta\,V(k',k) + \lambda\,[R\,k + w - c - k']$$

where $z' = k$. Controls are c, k'.

Alternative: Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1}) + \lambda_t\,[R_t k_t + w_t - c_t - k_{t+1}]$$

(b) First-order conditions from DP:

$$\begin{aligned} u_c(c,z) &= \lambda \\ \beta\, V_k(k',k) &= \lambda \end{aligned}$$

Envelope conditions:

$$\begin{aligned} V_k(k,z) &= \beta\, V_z(k',z') + \lambda\, R \\ V_z(k,z) &= u_z(c,z) \end{aligned}$$

Euler equation:

$$u_c(c,z) = \beta\, R'\, u_c(c',z') + \beta^2\, u_z(c'',z'')$$

Solution: Sequences (c_t, k_t) that solve the Euler equation, budget constraint, and transversality condition. Intuition: The additional u_z term in the Euler equation reflects the effect of raising k' in the usual perturbation.

Alternative: Lagrangian FOCs:

$$\begin{aligned} c_t &: \beta^t u_c\,(t) = \lambda_t \\ k_{t+1} &: \beta^{t+2} u_k\,(t+2) = \lambda_t + \lambda_{t+1} R_{t+1} \end{aligned}$$

(c) CE: Totally standard. $\{c_t, k_{t+1}, w_t, R_t\}$ that satisfy: Household Euler equation and budget constraint. 2 firm first-order conditions. Market clearing: $K_t = k_t$, $L_t = 1$, $F\,(K_t, L_t) + (1-\delta)\,K_t = c_t + K_{t+1}$.

(d) The Euler equation implies in steady state:

$$1 = \beta\,\left[f'(k) + 1 - \delta\right] + \beta^2\,\frac{u_z\,(f(k) - \delta\,k, k)}{u_c\,(f(k) - \delta\,k, k)}$$

(e) Steady state is generally not unique. Household may choose low c and high k or vice versa.

2 Ben-Porath Model

We study the decision problem of an infinitely lived agent in discrete time. At $t = 0$, the agent is endowed with h_0 units of human capital. In each period, he can invest l_t units of time, so that human capital evolves according to

$$h_{t+1} = (1-\delta)\,h_t + F\,(h_t l_t) \tag{2}$$

$$F\,(hl) = (hl)^{\alpha} \tag{3}$$

with $0 < \alpha, \delta < 1$. The objective is to maximize the present value of lifetime earnings, given by

$$Y = \sum_{t=0}^{\infty} R^{-t} w h_t\,(1 - l_t) \tag{4}$$

where $R > 0$ and $w > 0$ are taken as given.

Questions:

1. Write down the agent's Dynamic Program.
2. Derive and interpret the first-order condition for l.
3. Derive $V'(h) = w + (1-\delta) R^{-1} V'(h')$.
4. Derive and interpret $V'(h) = w\frac{R}{r+\delta}$ where $R = 1+r$.
5. How do the wage and the interest rate affect steady state h and l?

2.1 Answer: Ben-Porath Model[1]

1. $V(h) = \max_l wh(1-l) + R^{-1} V(h[1-\delta] + F(hl))$.
2. FOC: $hw = R^{-1} V'(h') F_l$. The left-hand size is the opportunity cost of studying. The right-hand-side is the marginal benefit: next period, the agent has additional human capital F_l, which is worth V'.
3. Substitute the first-order condition into the envelope equation

$$V'(h) = (1-l)w + R^{-1} V'(h')[1-\delta+F_h] \tag{5}$$

4. Iterate forward on $V'(h) = w + (1-\delta) R^{-1} V'(h')$. Note that

$$\sum_{j=0}^{\infty} ([1-\delta]/R)^j = \frac{R}{R-1+\delta} \tag{6}$$

   The value of human capital equals the present value earnings that it produces when the agent works full time. This is because, at the margin, the values of working and studying are the same.

5. The steady state satisfies $\delta h = F(hl)$ and

$$wh = R^{-1} F_l w \frac{R}{R-1+\delta} \tag{7}$$

   Solve these two conditions for

$$\bar{h} = \delta^{-1}\left(\frac{\alpha}{r+\delta}\right)^{\alpha/(1-\alpha)} \tag{8}$$

$$\bar{l} = \delta\left(\frac{\alpha}{r+\delta}\right) \tag{9}$$

   The wage does not affect $\bar{h}, \bar{l}$. The intuition is that the wage affects the marginal cost as much as the marginal benefit of studying. The interest rate reduces study time and hence human capital. The intuition is that it lowers the value of future earnings.

[1] Based on UC Davis qualifying exam 2016