Problem Set 3: Infinite Horizon Model in Discrete Time

Econ720. Fall 2019. Prof. Lutz Hendricks. October 15, 2019

1 Ben-Porath Model

We study the decision problem of an infinitely lived agent in discrete time. At $t = 0$, the agent is endowed with h_0 units of human capital. In each period, he can invest l_t units of time, so that human capital evolves according to

$$h_{t+1} = (1-\delta)\,h_t + F\,(h_t l_t) \tag{1}$$

$$F\,(hl) = (hl)^{\alpha} \tag{2}$$

with $0 < \alpha, \delta < 1$. The objective is to maximize the present value of lifetime earnings, given by

$$Y = \sum_{t=0}^{\infty} R^{-t} w h_t\,(1 - l_t) \tag{3}$$

where $R > 0$ and $w > 0$ are taken as given.

Questions:

1. Write down the agent's Dynamic Program.
2. Derive and interpret the first-order condition for l.
3. Derive $V'(h) = w + (1-\delta)\,R^{-1}V'(h')$.
4. Derive and interpret $V'(h) = w\frac{R}{r+\delta}$ where $R = 1 + r$.
5. How do the wage and the interest rate affect steady state h and l?

1.1 Answer: Ben-Porath Model[1]

1. $V(h) = \max_l wh\,(1-l) + R^{-1}V\,(h\,[1-\delta] + F\,(hl))$.
2. FOC: $hw = R^{-1}V'(h')\,F_l$. The left-hand size is the opportunity cost of studying. The right-hand-side is the marginal benefit: next period, the agent has additional human capital F_l, which is worth V'.
3. Substitute the first-order condition into the envelope equation

$$V'(h) = (1-l)\,w + R^{-1}V'(h')\,[1-\delta+F_h] \tag{4}$$

[1] Based on UC Davis qualifying exam 2016

4. Iterate forward on $V'(h) = w + (1-\delta) R^{-1} V'(h')$. Note that

$$\sum_{j=0}^{\infty} ([1-\delta]/R)^j = \frac{R}{R-1+\delta} \tag{5}$$

The value of human capital equals the present value earnings that it produces when the agent works full time. This is because, at the margin, the values of working and studying are the same.

5. The steady state satisfies $\delta h = F(hl)$ and

$$wh = R^{-1} F_l w \frac{R}{R-1+\delta} \tag{6}$$

Solve these two conditions for

$$\bar{h} = \delta^{-1} \left(\frac{\alpha}{r+\delta}\right)^{\alpha/(1-\alpha)} \tag{7}$$

$$\bar{l} = \delta \left(\frac{\alpha}{r+\delta}\right) \tag{8}$$

The wage does not affect $\bar{h}, \bar{l}$. The intuition is that the wage affects the marginal cost as much as the marginal benefit of studying. The interest rate reduces study time and hence human capital. The intuition is that it lowers the value of future earnings.

2 Education Costs

Consider the following version of a standard growth model with human capital. The planner solves

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t) \tag{9}$$

s.t.

$$k_{t+1} = (1-\delta) k_t + x_{kt} \tag{10}$$

$$h_{t+1} = (1-\delta) h_t + x_{ht} \tag{11}$$

$$c_t + x_{kt} + \eta x_{ht} = f(k_t, h_t) \tag{12}$$

with k_1 and h_1 given. Here c is consumption, k is physical capital, h is human capital, and η is a constant representing education costs. Assume that the production function is Cobb-Douglas:

$$f(k,h) = zk^{\alpha} h^{\varepsilon} \tag{13}$$

where z is a constant technology parameter and $\alpha + \varepsilon < 1$.

Questions:

1. Derive the first-order condition for the planner's problem using Dynamic Programming. Define a solution in sequence language and in functional language.

2. Solve for the steady state levels of k/h and k.

3. Characterize the impact of cross-country differences in education costs (η) on output per worker in steady state. In particular, calculate the ratio of outputs per worker for two countries that only differ in their η's.

2.1 Answer: Education Costs

(a) The planner's Bellman equation is

$$\begin{aligned} V(k,h) &= \max u(c) + \beta V((1-\delta)k + x_k, (1-\delta)h + x_h) \\ &\quad +\lambda[f(k,h) - c - x_k - \eta x_h - g] \end{aligned}$$

First-order conditions:

$$\begin{aligned} u'(c) &= \lambda \\ \beta V_k(.') &= \lambda \\ \beta V_h(.') &= \eta\lambda \end{aligned}$$

Envelope conditions:

$$\begin{aligned} V_k(k,h) &= \beta V_k(k',h')(1-\delta) + \lambda f_k(k,h) \\ V_h(k,h) &= \beta V_h(k',h')(1-\delta) + \lambda f_h(k,h) \end{aligned}$$

Simplify to obtain an Euler equation, which is perfectly standard:

$$u'(c) = \beta u'(c')[1-\delta + f_k(k',h')]$$

In addition, there is a second Euler equation

$$u'(c) = \beta u'(c')[1-\delta + f_h(k',h')/\eta]$$

which can be made into a static condition

$$1-\delta + f_k(k',h') = 1-\delta + f_h(k',h')/\eta$$

A solution consists of sequences c, k, h, x_k, x_h that solve 2 laws of motion, 1 feasibility condition, 2 first-order conditions.

(b) Imposing functional forms: $k/h = \eta\alpha/\varepsilon$. The steady state capital stock is determined by

$$1/\beta = z\alpha k^{\alpha-1+\varepsilon}[\varepsilon/(\alpha\eta)]^{\varepsilon} + 1 - \delta$$

Steady state output is

$$f(k_{ss}, h_{ss}) = z k_{ss}^{\alpha+\varepsilon}[\varepsilon/(\alpha\eta)]^{\varepsilon}$$

(c) An increase in η reduces both k and h in steady state. How much do education costs affect output per worker? The output ratio of two countries is

$$\frac{f^A}{f^B} = \left(\frac{k_{ss}^A}{k_{ss}^B}\right)^{\alpha+\varepsilon} \left(\frac{\eta_B}{\eta_A}\right)^{\varepsilon}$$

The ratio of capital stocks can be derived from the steady state k equation:

$$k_{ss}^A/k_{ss}^B = (\eta_A/\eta_B)^{\varepsilon/(\alpha+\varepsilon-1)}$$

Finally,

$$f^A/f^B = (\eta_A/\eta_B)^{\varepsilon/(1-\alpha-\varepsilon)}$$