

# Dynamic Programming Theorems

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# Dynamic Programming Theorems

Useful theorems to characterize the solution to a DP problem.

There is no reason to remember these results.

But you need to know they exist and can be looked up when you need them.

## Generic Sequence Problem (P1)

$$\begin{aligned} V^*(x(0)) &= \max_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\ &\text{subject to} \\ x(t+1) &\in G(x(t)) \\ x(0) &\text{ given} \end{aligned}$$

$x(t) \in X \subset \mathbb{R}^k$  is the set of allowed states.

The correspondence  $G: X \rightrightarrows X$  defines the constraints.

A solution is a sequence  $\{x(t)\}$

## Mapping into the growth model

$$\max_{\{k(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k(t)) - k(t+1))$$

subject to

$$k(t+1) \in G(k(t)) = [0, f(k(t))]$$

$$k(t) \in X = \mathbb{R}^+$$

$$k(0) \text{ given}$$

## Recursive Problem (P2)

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

A solution is a policy function  $\pi : X \rightarrow X$  and a value function  $V(x)$  such that

1.  $V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \quad \forall x \in X$
2. When  $y = \pi(x)$ , now and forever, the max value is attained.

# The Main Point

This is the upshot of everything that follows:

*If it is possible to write the optimization problem in the **format** of  $P1$  and if mild **conditions** hold, then solving  $P1$  and  $P2$  are **equivalent**.*

# Assumptions That Could Be Relaxed

1. Stationarity:  $U$  and  $G$  do not depend on  $t$ .
2. Utility is additively separable.
  - ▶ Time consistency
3. The control is  $x(t+1)$ .
  - ▶ There could be additional controls that don't affect  $x(t+1)$ .
  - ▶ They are "max'd out". Ex: 2 consumption goods.

# Dynamic Programming Theorems

- ▶ The payoff of DP: it is easier to prove that solutions exist, are unique, monotone, etc.
- ▶ We state some assumptions and theorems using them.



## Assumption 1: Non-emptiness

- ▶ Define the set of feasible paths starting at  $x(0)$  by  $\Phi(x(0))$ .
- ▶  $G(x)$  is **nonempty** for all  $x \in X$ .
  - ▶ needed to prevent a currently good looking path from running into "dead ends"
- ▶  $\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t U(x(t), x(t+1))$  exists and is **finite**, for all  $x(0) \in X$  and feasible paths  $\mathbf{x} \in \Phi(x(0))$ .
  - ▶ cannot have unbounded utility

## Assumption 2: Compactness

- ▶ The set  $X$  in which  $x$  lives is **compact**.
- ▶  $G$  is compact valued and continuous.
- ▶  $U$  is continuous.

### Notes:

- ▶ Compactness avoids existence issues: without it, there could always be a slightly better  $x$
- ▶ Compact  $X$  creates trouble with endogenous growth, but can be relaxed.
- ▶ Think of A1 and A2 together as the “existence conditions.”

## Digression

A correspondence is **continuous** if it has a closed graph and is lower hemicontinuous.

**Closed graph:**  $\{x_n\} \rightarrow x$  and  $\{y_n\} \rightarrow y \implies y \in G(x)$  where

▶  $G : X \rightrightarrows Y, x_n \in X, y_n \in Y.$

**Lower hemicontinuity:**

For every  $\{x_n\} \rightarrow x \in X$  there is a  $y_n$  that satisfies

1.  $y_n \in G(x_n)$
2.  $\{y_n\} \rightarrow y \in G(x)$

## Assumption 3: Convexity

- ▶  $U$  is strictly concave.
- ▶  $G$  is convex (for all  $x$ ,  $G(x)$  is a convex set).

Typical assumptions to ensure that **first order conditions** are sufficient.

## Assumption 4: Monotonicity

- ▶  $U(x,y)$  is strictly increasing in  $x$ .
  - ▶ more capital is better
- ▶  $G$  is monotone in the sense that  $x \leq x'$  implies  $G(x) \subset G(x')$ .

This is needed for **monotonicity** of policy function.

## Assumption 5: Differentiability

- ▶  $U$  is continuously **differentiable** on the interior of its domain.

So we can work with first-order conditions.

# Main Result

**Principle of Optimality** + Equivalence of values:

A1 and A2  $\implies$  solving P1 and solving P2 yield the same value and policy functions.

You can read about the details...

## Theorem 3: Uniqueness of $V$

- ▶ Assumptions: A1 and A2.
- ▶ Then there exists a unique, **continuous, bounded** value function that solves P1 or P2 (they are the same).
- ▶ An optimal plan  $\mathbf{x}^*$  exists. But it may not be unique.



## Theorem 4: Concavity of $V$

- ▶ Assumptions: A1-A3 (convexity).
- ▶ Then the value function is strictly concave.

Recall: A3 says that  $U$  is strictly concave and  $G(x)$  is convex.  
So we are solving a concave / convex programming problem.

## Corollary 1

- ▶ Assumptions A1-A3.
- ▶ Then there exists a **unique optimal plan**  $\mathbf{x}^*$  for all  $x(0)$ .
- ▶ It can be written as  $x^*(t+1) = \pi(x^*(t))$ .
- ▶  $\pi$  is continuous.

Reason: The Bellman equation is a concave optimization problem with convex choice set.

## Theorem 5: Monotonicity of $V$

- ▶ Assumptions: A1, A2, A4.
- ▶ Recall A4:  $U$  and  $G$  are monotone.
- ▶  $V$  is strictly increasing in all arguments (states).

## Theorem 6: Differentiability of $V$

- ▶ Assumptions A1, A2, A3, A5.
- ▶ A5:  $U$  is differentiable.
- ▶ Then  $V(x)$  is continuously differentiable at all interior points  $x'$  with  $\pi(x') \in \text{Int}G(x')$ .
- ▶ The derivative is given by:

$$DV(x') = D_x U(x', \pi(x')) \quad (1)$$

This is an envelope condition: we can ignore the response of  $\pi$  when  $x'$  changes.

## Contraction mapping theorem

- ▶ How could one show that  $V$  is increasing? Or concave? Etc.
- ▶ Thinking of the Bellman equation as a functional equation helps...
- ▶ Think of the Bellman equation as mapping  $V$  on the RHS into  $\hat{V}$  on the LHS:

$$\hat{V}(x) = \max_{y \in G(x)} U(x, y) + \beta V(y) \quad (2)$$

- ▶ The RHS is a function of  $V$ .
- ▶ The Bellman equation maps the space of functions  $V$  lives in into itself.

$$\hat{V} = T(V) \quad (3)$$

- ▶ The solution is the function  $V$  that is a fixed point of  $T$ :

$$V = T(V) \quad (4)$$

# Notation

- ▶ If  $T : X \rightarrow X$ , we write:
  1.  $Tx$  instead of the usual  $T(x)$
  2.  $T(\hat{X})$  as the image of the set  $\hat{X} \subset X$ .

## Contraction mapping theorem

- ▶ The Bellman equation is  $\hat{V} = TV$ .
- ▶ Suppose we could show:
  1. If  $V$  is increasing, then  $\hat{V}$  is increasing.
  2. There is a fixed point in the set of increasing functions.
  3. The fixed point is unique.
- ▶ Then we would have shown that the solution  $V$  is increasing.
- ▶ The contraction mapping theorem allows us to make arguments like this.

# Contraction mapping theorem

## Definition

Let  $(S, d)$  be a metric space and  $T : S \rightarrow S$ .  $T$  is a contraction mapping with modulus  $\beta$ , if for some  $\beta \in (0, 1)$ ,

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \quad \forall z_1, z_2 \in S \quad (5)$$

A contraction pulls points closer together.



# Contraction mapping theorem

*Theorem 7: Let  $(S, d)$  be a complete metric space and let  $T$  be a contraction mapping. Then  $T$  has a unique fixed point in  $S$ .*

Recall:

1. Cauchy sequence: For any  $\varepsilon$ ,  $\exists n$  such that  $d(x_n, x_m) < \varepsilon$  for  $m > n$ .
2. Complete metric space: Every Cauchy sequence converges to a point in  $S$ .

## Contraction mapping theorem

A helpful result for showing properties of  $V$  :

*Theorem 8: Let  $(S, d)$  be a complete metric space and let  $T : S \rightarrow S$  be a contraction mapping with fixed point  $T\hat{z} = \hat{z}$ .*

*If  $S'$  is a closed subset of  $S$  and  $T(S') \subset S'$ , then  $\hat{z} \in S'$ .*

*If  $T(S') \subset S'' \subset S'$ , then  $\hat{z} \in S''$ .*

The point: When looking for the fixed point, one can restrict the search to sub-spaces with nice properties.

Example:

- ▶ We try to show that  $V$  is strictly concave, but the set of strictly concave functions  $(S)$  is not closed.
- ▶ If we can show that  $T$  maps strictly concave functions into a closed subset  $S'$  of  $S$ , then  $V$  must be strictly concave.

## Blackwell's Sufficient Conditions

This is helpful for showing that a Bellman operator is a contraction:

*Theorem 9: Let  $X \subseteq \mathbb{R}^K$ , and  $\mathbf{B}(X)$  be the space of bounded functions  $f : X \rightarrow \mathbb{R}$ . Suppose that  $T : \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  satisfies:*

*(1) monotonicity:  $f(x) \leq g(x)$  for all  $x \in X$  implies  $Tf(x) \leq Tg(x)$  for all  $x \in X$ .*

*(2) discounting: there exists  $\beta \in (0, 1)$  such that*

$$T[f(x) + c] \leq Tf(x) + \beta c \text{ for all } f \in \mathbf{B}(X) \text{ and } c \geq 0.$$

*Then  $T$  is a contraction with modulus  $\beta$ .*

## Example: Growth Model

$$TV = \max_{k' \in [0, f(k)]} U(f(k) - k') + \beta V(k') \quad (6)$$

Metric space:

- ▶  $S$ : set of bounded functions on  $(0, \infty)$ .
- ▶  $d$ : sup norm:  $d(f, g) = \sup |f(k) - g(k)|$ .

Step 1:  $T : S \rightarrow S$

- ▶ need tricks if  $U$  is not bounded (argue that  $k$  is bounded along any feasible path)
- ▶ otherwise  $TV$  is the sum of bounded functions

## Example: Growth Model

### Step 2: Monotonicity

- ▶ Assume  $W(k) \geq V(k) \forall k$ .
- ▶ Let  $g(k)$  be the optimal policy for  $V(k)$ .
- ▶ Then

$$TV(k) = U(f(k) - g(k)) + \beta V(g(k)) \quad (7)$$

$$\leq U(f(k) - g(k)) + \beta W(g(k)) \quad (8)$$

$$\leq TW(k) \quad (9)$$

## Example: Growth Model

Step 3: Discounting

$$T(V + a(k)) = \max U(f(k) - k') + \beta[V(k') + a] \quad (10)$$

$$= V(k) + \beta a \quad (11)$$

Therefore:  $T$  is a contraction mapping with modulus  $\beta$ .

## Summary: Contraction mapping theorem

Suppose you want to show that the value function is increasing.

1. Show that the Bellman equation is a contraction mapping - using Blackwell.
2. Show that it maps increasing functions into increasing functions.

Done.

## First order conditions

Consider again Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

If we make assumptions that ensure:

- ▶  $V$  is differentiable and concave.
- ▶  $U$  is concave.
- ▶  $G$  is convex. [A1-A5 ensure all that.]

Then the RHS is just a standard concave optimization problem.

We can take the usual FOCs to characterize the solution.



## First order conditions

- ▶ For  $y$ :

$$D_y U(x, \pi(x)) + \beta DV(\pi(x)) = 0 \quad (12)$$

- ▶ To find  $DV(x)$  differentiate the Bellman equation:

$$DV(x) = D_x U(x, \pi(x)) + D_y U(x, \pi(x)) D\pi(x) + \beta DV(\pi(x)) D\pi(x) = \quad (13)$$

- ▶ Apply the FOC to find the Envelope condition:

$$DV(x) = D_x U(x, \pi(x)) \quad (14)$$

$$DV(\pi(x)) = D_x U(\pi(x), \pi(\pi(x))) \quad (15)$$

- ▶ Sub back into the FOC:

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0 \quad (16)$$

## First order conditions

- ▶ In the usual prime notation:

$$D_2U(x, x') + \beta D_1U(x', x'') = 0 \quad (17)$$

- ▶ Think about a feasible perturbation:
  1. Raise  $x'$  a little and gain  $D_2U(x, x')$  today.
  2. Tomorrow lose the marginal value of the state  $x'$ :  $D_1(x', x'')$ .
- ▶ Why isn't there a term as in the growth model's resource constraint:  $f'(k) + 1 - \delta$ ?
  - ▶ By writing  $U(x, x')$ , the resource constraint is built into  $U$ .
  - ▶ In the growth model:  $U(k, k') = u(f(k) + (1 - \delta)k - k')$ .
  - ▶  $D_1U = u'(c)[f'(k) + 1 - \delta]$ .

# Transversality

- ▶ Even though the programming problem is concave, the first-order condition is not sufficient!
- ▶ A mechanical reason: it is a first-order difference equation - it has infinitely many solutions.
- ▶ A boundary condition is needed.

*Theorem 10: Let  $X \subset \mathbb{R}^K$  and assume A1-A5. Then a sequence  $\{x(t+1)\}$  with  $x(t+1) \in \text{Int}G(x(t))$  is optimal in P1, if it satisfies the Euler equation and the transversality condition*

$$\lim_{t \rightarrow \infty} \beta^t D_x U(x(t), x(t+1)) x(t) = 0 \quad (18)$$

## Example: The growth model

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c(t))$$

*subject to*

$$0 \leq k(t+1) \leq k(t)^\alpha - c(t)$$

$$k(0) = k_0$$

## Example: The growth model

- ▶ Step 1: Show that A1 to A5 hold.
- ▶ Define  $U(k, k') = \ln(k^\alpha - k')$ .
- ▶ A1 is obvious:  $G(x)$  is non-empty. The sum of discounted utilities is bounded for all feasible paths.
- ▶ A2:
  - ▶  $X$  is compact - no, but we can restrict  $k$  to a compact set w.l.o.g.
  - ▶  $G$  is compact valued and continuous: check
  - ▶  $U$  is continuous: check
- ▶ A3:  $U$  is strictly concave.  $G(x)$  is convex: check.
- ▶ A4:  $U$  is strictly increasing in  $x$ .  $G$  is monotone: check.
- ▶ A5:  $U$  is continuously differentiable: check

## Example: The growth model

- ▶ Step 2: Theorems 1-6 and 10 apply.
- ▶ We can characterize the solution by first-order conditions and TVC.
- ▶ FOC:

$$\frac{1}{k^\alpha - \pi(k)} = \beta V'(\pi(k)) \quad (19)$$

- ▶ Envelope:

$$V'(k) = \frac{\alpha k^{\alpha-1}}{k^\alpha - \pi(k)} \quad (20)$$

- ▶ Combine:

$$\frac{1}{k^\alpha - \pi(k)} = \beta \frac{\alpha \pi(k)^{\alpha-1}}{\pi(k)^\alpha - \pi(\pi(k))} \quad (21)$$

- ▶ Or:

$$u'(c) = \beta f'(k') u'(c') \quad (22)$$

## Example: The growth model

Other things we know:

1.  $V$  is continuously differentiable, bounded, unique, strictly concave.
2.  $V'(k) > 0$ .
3. The optimal policy function  $c = \phi(k)$  is unique, continuous.

## Reading

- ▶ Acemoglu, *Introduction to Modern Economic Growth*, ch. 6
- ▶ Stokey, Lucas, with Prescott, *Recursive Methods*. A book length treatment. The standard reference.
- ▶ Krusell, “Real Macroeconomic Theory,” ch. 4.