

Dynamic Programming Theorems

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Dynamic Programming Theorems

Useful theorems to characterize the solution to a DP problem.

There is no reason to remember these results.

But you need to know they exist and can be looked up when you need them.

Generic Sequence Problem (P1)

$$\begin{aligned} V^*(x(0)) &= \max_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\ &\text{subject to} \\ x(t+1) &\in G(x(t)) \\ x(0) &\text{ given} \end{aligned}$$

$x(t) \in X \subset \mathbb{R}^k$ is the set of allowed states.

The correspondence $G: X \rightrightarrows X$ defines the constraints.

A solution is a sequence $\{x(t)\}$

Mapping into the growth model

$$\max_{\{k(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k(t)) - k(t+1))$$

subject to

$$k(t+1) \in G(k(t)) = [0, f(k(t))]$$

$$k(t) \in X = \mathbb{R}^+$$

$$k(0) \text{ given}$$

Recursive Problem (P2)

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

A solution is a policy function $\pi : X \rightarrow X$ and a value function $V(x)$ such that

1. $V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \quad \forall x \in X$
2. When $y = \pi(x)$, now and forever, the max value is attained.

The Main Point

This is the upshot of everything that follows:

*If it is possible to write the optimization problem in the **format** of $P1$
and if mild **conditions** hold,
then solving $P1$ and $P2$ are **equivalent**.*

Assumptions That Could Be Relaxed

1. Stationarity: U and G do not depend on t .
2. Utility is additively separable.
 - ▶ Time consistency
3. The control is $x(t+1)$.
 - ▶ There could be additional controls that don't affect $x(t+1)$.
 - ▶ They are "max'd out". Ex: 2 consumption goods.

Dynamic Programming Theorems

- ▶ The payoff of DP: it is easier to prove that solutions exist, are unique, monotone, etc.
- ▶ We state some assumptions and theorems using them.

Assumption 1: Non-emptiness

- ▶ Define the set of feasible paths starting at $x(0)$ by $\Phi(x(0))$.
- ▶ $G(x)$ is **nonempty** for all $x \in X$.
 - ▶ needed to prevent a currently good looking path from running into "dead ends"
- ▶ $\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t U(x(t), x(t+1))$ exists and is **finite**, for all $x(0) \in X$ and feasible paths $\mathbf{x} \in \Phi(x(0))$.
 - ▶ cannot have unbounded utility

Assumption 2: Compactness

- ▶ The set X in which x lives is **compact**.
- ▶ G is compact valued and continuous.
- ▶ U is continuous.

Notes:

- ▶ Compactness avoids existence issues: without it, there could always be a slightly better x
- ▶ Compact X creates trouble with endogenous growth, but can be relaxed.
- ▶ Think of A1 and A2 together as the “existence conditions.”

Digression

A correspondence is **continuous** if it has a closed graph and is lower hemicontinuous.

Closed graph: $\{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y \implies y \in G(x)$ where

▶ $G : X \rightrightarrows Y, x_n \in X, y_n \in Y.$

Lower hemicontinuity:

For every $\{x_n\} \rightarrow x \in X$ there is a y_n that satisfies

1. $y_n \in G(x_n)$
2. $\{y_n\} \rightarrow y \in G(x)$

Assumption 3: Convexity

- ▶ U is strictly concave.
- ▶ G is convex (for all x , $G(x)$ is a convex set).

Typical assumptions to ensure that **first order conditions** are sufficient.

Assumption 4: Monotonicity

- ▶ $U(x,y)$ is strictly increasing in x .
 - ▶ more capital is better
- ▶ G is monotone in the sense that $x \leq x'$ implies $G(x) \subset G(x')$.

This is needed for **monotonicity** of policy function.

Assumption 5: Differentiability

- ▶ U is continuously **differentiable** on the interior of its domain.

So we can work with first-order conditions.

Main Result

Principle of Optimality + Equivalence of values:

$A1$ and $A2 \implies$ solving $P1$ and solving $P2$ yield the same value and policy functions.

You can read about the details...

Theorem 3: Uniqueness of V

- ▶ Assumptions: A1 and A2.
- ▶ Then there exists a unique, **continuous, bounded** value function that solves P1 or P2 (they are the same).
- ▶ An optimal plan \mathbf{x}^* exists. But it may not be unique.

Theorem 4: Concavity of V

- ▶ Assumptions: A1-A3 (convexity).
- ▶ Then the value function is strictly concave.

Recall: A3 says that U is strictly concave and $G(x)$ is convex.
So we are solving a concave / convex programming problem.

Corollary 1

- ▶ Assumptions A1-A3.
- ▶ Then there exists a **unique optimal plan** \mathbf{x}^* for all $x(0)$.
- ▶ It can be written as $x^*(t+1) = \pi(x^*(t))$.
- ▶ π is continuous.

Reason: The Bellman equation is a concave optimization problem with convex choice set.

Theorem 5: Monotonicity of V

- ▶ Assumptions: A1, A2, A4.
- ▶ Recall A4: U and G are monotone.
- ▶ V is strictly increasing in all arguments (states).

Theorem 6: Differentiability of V

- ▶ Assumptions A1, A2, A3, A5.
- ▶ A5: U is differentiable.
- ▶ Then $V(x)$ is continuously differentiable at all interior points x' with $\pi(x') \in \text{Int}G(x')$.
- ▶ The derivative is given by:

$$DV(x') = D_x U(x', \pi(x')) \quad (1)$$

This is an envelope condition: we can ignore the response of π when x' changes.

Contraction mapping theorem

- ▶ How could one show that V is increasing? Or concave? Etc.
- ▶ Thinking of the Bellman equation as a functional equation helps...
- ▶ Think of the Bellman equation as mapping V on the RHS into \hat{V} on the LHS:

$$\hat{V}(x) = \max_{y \in G(x)} U(x, y) + \beta V(y) \quad (2)$$

- ▶ The RHS is a function of V .
- ▶ The Bellman equation maps the space of functions V lives in into itself.

$$\hat{V} = T(V) \quad (3)$$

- ▶ The solution is the function V that is a fixed point of T :

$$V = T(V) \quad (4)$$

Notation

- ▶ If $T : X \rightarrow X$, we write:
 1. Tx instead of the usual $T(x)$
 2. $T(\hat{X})$ as the image of the set $\hat{X} \subset X$.

Contraction mapping theorem

- ▶ The Bellman equation is $\hat{V} = TV$.
- ▶ Suppose we could show:
 1. If V is increasing, then \hat{V} is increasing.
 2. There is a fixed point in the set of increasing functions.
 3. The fixed point is unique.
- ▶ Then we would have shown that the solution V is increasing.
- ▶ The contraction mapping theorem allows us to make arguments like this.

Contraction mapping theorem

Definition

Let (S, d) be a metric space and $T : S \rightarrow S$. T is a contraction mapping with modulus β , if for some $\beta \in (0, 1)$,

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \quad \forall z_1, z_2 \in S \quad (5)$$

A contraction pulls points closer together.

Contraction mapping theorem

Theorem 7: Let (S, d) be a complete metric space and let T be a contraction mapping. Then T has a unique fixed point in S .

Recall:

1. Cauchy sequence: For any ε , $\exists n$ such that $d(x_n, x_m) < \varepsilon$ for $m > n$.
2. Complete metric space: Every Cauchy sequence converges to a point in S .

Contraction mapping theorem

A helpful result for showing properties of V :

Theorem 8: Let (S, d) be a complete metric space and let $T : S \rightarrow S$ be a contraction mapping with fixed point $T\hat{z} = \hat{z}$.

If S' is a closed subset of S and $T(S') \subset S'$, then $\hat{z} \in S'$.

If $T(S') \subset S'' \subset S'$, then $\hat{z} \in S''$.

The point: When looking for the fixed point, one can restrict the search to sub-spaces with nice properties.

Example:

- ▶ We try to show that V is strictly concave, but the set of strictly concave functions (S) is not closed.
- ▶ If we can show that T maps strictly concave functions into a closed subset S' of S , then V must be strictly concave.

Blackwell's Sufficient Conditions

This is helpful for showing that a Bellman operator is a contraction:

Theorem 9: Let $X \subseteq \mathbb{R}^K$, and $\mathbf{B}(X)$ be the space of bounded functions $f : X \rightarrow \mathbb{R}$. Suppose that $T : \mathbf{B}(X) \rightarrow \mathbf{B}(X)$ satisfies:

(1) monotonicity: $f(x) \leq g(x)$ for all $x \in X$ implies $Tf(x) \leq Tg(x)$ for all $x \in X$.

(2) discounting: there exists $\beta \in (0, 1)$ such that

$$T[f(x) + c] \leq Tf(x) + \beta c \text{ for all } f \in \mathbf{B}(X) \text{ and } c \geq 0.$$

Then T is a contraction with modulus β .

Example: Growth Model

$$TV = \max_{k' \in [0, f(k)]} U(f(k) - k') + \beta V(k') \quad (6)$$

Metric space:

- ▶ S : set of bounded functions on $(0, \infty)$.
- ▶ d : sup norm: $d(f, g) = \sup |f(k) - g(k)|$.

Step 1: $T : S \rightarrow S$

- ▶ need tricks if U is not bounded (argue that k is bounded along any feasible path)
- ▶ otherwise TV is the sum of bounded functions

Example: Growth Model

Step 2: Monotonicity

- ▶ Assume $W(k) \geq V(k) \forall k$.
- ▶ Let $g(k)$ be the optimal policy for $V(k)$.
- ▶ Then

$$TV(k) = U(f(k) - g(k)) + \beta V(g(k)) \quad (7)$$

$$\leq U(f(k) - g(k)) + \beta W(g(k)) \quad (8)$$

$$\leq TW(k) \quad (9)$$

Example: Growth Model

Step 3: Discounting

$$T(V + a(k)) = \max U (f(k) - k') + \beta[V(k') + a] \quad (10)$$

$$= V(k) + \beta a \quad (11)$$

Therefore: T is a contraction mapping with modulus β .

Summary: Contraction mapping theorem

Suppose you want to show that the value function is increasing.

1. Show that the Bellman equation is a contraction mapping - using Blackwell.
2. Show that it maps increasing functions into increasing functions.

Done.

First order conditions

Consider again Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

If we make assumptions that ensure:

- ▶ V is differentiable and concave.
- ▶ U is concave.
- ▶ G is convex. [A1-A5 ensure all that.]

Then the RHS is just a standard concave optimization problem.

We can take the usual FOCs to characterize the solution.

First order conditions

- ▶ For y :

$$D_y U(x, \pi(x)) + \beta DV(\pi(x)) = 0 \quad (12)$$

- ▶ To find $DV(x)$ differentiate the Bellman equation:

$$DV(x) = D_x U(x, \pi(x)) + D_y U(x, \pi(x)) D\pi(x) + \beta DV(\pi(x)) D\pi(x) = 0 \quad (13)$$

- ▶ Apply the FOC to find the Envelope condition:

$$DV(x) = D_x U(x, \pi(x)) \quad (14)$$

$$DV(\pi(x)) = D_x U(\pi(x), \pi(\pi(x))) \quad (15)$$

- ▶ Sub back into the FOC:

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0 \quad (16)$$

First order conditions

- ▶ In the usual prime notation:

$$D_2U(x, x') + \beta D_1U(x', x'') = 0 \quad (17)$$

- ▶ Think about a feasible perturbation:
 1. Raise x' a little and gain $D_2U(x, x')$ today.
 2. Tomorrow lose the marginal value of the state x' : $D_1(x', x'')$.
- ▶ Why isn't there a term as in the growth model's resource constraint: $f'(k) + 1 - \delta$?
 - ▶ By writing $U(x, x')$, the resource constraint is built into U .
 - ▶ In the growth model: $U(k, k') = u(f(k) + (1 - \delta)k - k')$.
 - ▶ $D_1U = u'(c)[f'(k) + 1 - \delta]$.

Transversality

- ▶ Even though the programming problem is concave, the first-order condition is not sufficient!
- ▶ A mechanical reason: it is a first-order difference equation - it has infinitely many solutions.
- ▶ A boundary condition is needed.

Theorem 10: Let $X \subset \mathbb{R}^K$ and assume A1-A5. Then a sequence $\{x(t+1)\}$ with $x(t+1) \in \text{Int}G(x(t))$ is optimal in P1, if it satisfies the Euler equation and the transversality condition

$$\lim_{t \rightarrow \infty} \beta^t D_x U(x(t), x(t+1)) x(t) = 0 \quad (18)$$

Example: The growth model

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c(t))$$

subject to

$$0 \leq k(t+1) \leq k(t)^\alpha - c(t)$$

$$k(0) = k_0$$

Example: The growth model

- ▶ Step 1: Show that A1 to A5 hold.
- ▶ Define $U(k, k') = \ln(k^\alpha - k')$.
- ▶ A1 is obvious: $G(x)$ is non-empty. The sum of discounted utilities is bounded for all feasible paths.
- ▶ A2:
 - ▶ X is compact - no, but we can restrict k to a compact set w.l.o.g.
 - ▶ G is compact valued and continuous: check
 - ▶ U is continuous: check
- ▶ A3: U is strictly concave. $G(x)$ is convex: check.
- ▶ A4: U is strictly increasing in x . G is monotone: check.
- ▶ A5: U is continuously differentiable: check

Example: The growth model

- ▶ Step 2: Theorems 1-6 and 10 apply.
- ▶ We can characterize the solution by first-order conditions and TVC.
- ▶ FOC:

$$\frac{1}{k^\alpha - \pi(k)} = \beta V'(\pi(k)) \quad (19)$$

- ▶ Envelope:

$$V'(k) = \frac{\alpha k^{\alpha-1}}{k^\alpha - \pi(k)} \quad (20)$$

- ▶ Combine:

$$\frac{1}{k^\alpha - \pi(k)} = \beta \frac{\alpha \pi(k)^{\alpha-1}}{\pi(k)^\alpha - \pi(\pi(k))} \quad (21)$$

- ▶ Or:

$$u'(c) = \beta f'(k') u'(c') \quad (22)$$

Example: The growth model

Other things we know:

1. V is continuously differentiable, bounded, unique, strictly concave.
2. $V'(k) > 0$.
3. The optimal policy function $c = \phi(k)$ is unique, continuous.

Reading

- ▶ Acemoglu, *Introduction to Modern Economic Growth*, ch. 6
- ▶ Stokey, Lucas, with Prescott, *Recursive Methods*. A book length treatment. The standard reference.
- ▶ Krusell, “Real Macroeconomic Theory,” ch. 4.