Cash-in-Advance Model

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1. Introduction

We study a second model of money.

Models where money is a bubble (such as the OLG model we studied) have 2 shortcomings:

- They fail to explain rate of return dominance.
- Money has no transaction value.

CIA models focus on **transactions demand** for money. The basic idea:

- There is a **technology** that produces liquidity services.
- Buying certain goods requires those services.
- ▶ The simplest case: only cash can be used to buy those goods.

2. Environment

Demographics:

- a representative household of mass 1
- no firms; households operate the technology
- Preferences: $\sum_{t=1}^{\infty} \beta^t u(c_t)$
- Endowments at t = 1:
 - m_{t-1}^d units of money;
 - k₁ units of the good

Technologies:

• $f(k_t) + (1 - \delta)k_t = c_t + k_{t+1}$

Environment

Government

- Costlessly prints τ_t units of money and hands it to households (lump-sum)
- The nominal money growth rule is

 $\tau_t = g \times M_{t-1}$

g > 0 is a parameter

Markets:

- goods: price pt
- money: price 1

Notation:

- m is what the household chooses
- M is aggregate money
- in equilibrium, they are the same

Cash-in-advance constraint

Transactions technology:

$$m_t/p_t \ge c_t + k_{t+1} - (1 - \delta)k_t$$
 (1)

Requires that some goods are purchased with money.

An odd feature:

- the CIA constraint really is a technology
- its output: transactions services
- its input can be produced at no cost

Money

At the start of *t*:

► households hold m_{t-1}^d

• the government hands out τ_t (lump sum)

Households now hold $m_t = m_{t-1}^d + \tau_t$

- This can be used to buy things in t
- But no money changes hands (just have to have in your pocket).

Purchased money is taken into t + 1: m_t^d

Note that money earned in period t cannot be used until t+1.

3. Household

Budget constraint:

$$k_{t+1} + c_t + m_t^d/p_t = f(k_t) + (1 - \delta)k_t + m_t/p_t$$

Savings are taken into the next period in the form of capital and money $% \left({{{\left[{{{\rm{T}}_{\rm{T}}} \right]}}} \right)$

Household problem

We simply add one constraint to the household problem: the CIA constraint.

The household solves

 $\max \sum_{t=1}^{\infty} \beta^t u(c_t)$

subject to the budget constraint

$$k_{t+1} + c_t + m_t^d/p_t = f(k_t) + (1 - \delta)k_t + m_t/p_t$$

and the CIA constraint

$$m_t/p_t \geq c_t + k_{t+1} - (1-\delta)k_t$$

and the law of motion

$$m_{t+1} = m_t^d + \tau_{t+1}$$

Household problem Remarks

- Exactly what kinds of goods have to be bought with cash is arbitrary.
- The CIA constraint holds with equality if the rate of return on money is less than that on capital (the nominal interest rate is positive).

3.1. Household: Dynamic Program

Individual state variables: *m*,*k*. Bellman equation:

$$V(m,k) = \max u(c) + \beta V(m',k') + \lambda (BC) + \gamma (CIA)$$

We need to impose

$$m_t = m_{t-1}^d + \tau_t$$

Then we can use m_{t+1} as a control (this would not work under uncertainty).

Bellman Equation

$$V(m,k) = \max u(c) + \beta V(m',k') + \lambda [f(k) + (1-\delta)k + m/p - c - k' - (m' - \tau')/p] + \gamma [m/p - c - k' + (1-\delta)k]$$

 $\lambda > 0$: multiplier on budget constraint γ : multiplier on CIA constraint - could be 0.

Notes:

- We could substitute the budget constraint into u(c), but it's instructive not to.
- We cannot sub out the CIA constraint (why not?)

First-order conditions

$$u'(c) = \lambda + \gamma$$

 $\beta V_m(\bullet') = \lambda/p$
 $\beta V_k(\bullet') = \lambda + \gamma$

Envelope conditions:

$$V_m = (\lambda + \gamma)/p$$

$$V_k = \lambda [f'(k) + 1 - \delta] + \gamma [1 - \delta]$$

Interpretation ...

Simplify

Simplify (eliminate V's and $\lambda + \gamma$'s):

$$u'(c) / \beta = \lambda' f'(k') + [1 - \delta] u'(c')$$

$$\beta u'(c') p / p' = \lambda$$

$$u'(c) = \lambda + \gamma$$

Interpretation...

Kuhn Tucker:

$$\gamma[m/p-c-k'+(1-\delta)k] = 0$$

 $\gamma \geq 0$

Household: Solution

A solution to the household problem: $\{c_t, m_{t+1}, k_{t+1}, \lambda_t, \gamma_t\}$ that solve

- 1. 3 FOCs
- 2. budget constraint
- 3. either CIA constraint or $\gamma = 0$
- 4. transversality conditions

$$\lim_{t\to\infty} \beta^t u'(c_t) (k_t + m_t/p_t) = 0$$
(2)

Note: There is one TVC for the total value of assets.

to see why, think about red and green capital...

3.2. Household: CIA does not bind With $\gamma = 0$:

$$\begin{array}{rcl} \beta\lambda'/p' &=& \lambda/p \\ \lambda/\beta &=& \lambda'[f'(k')+1-\delta] \\ u'(c) &=& \lambda \end{array}$$

Standard Euler equation:

$$u'(c) = \beta u'(c') [f'(k') + 1 - \delta]$$
 (3)

"No arbitrage" condition:

$$f'(k') + 1 - \delta = p/p' \tag{4}$$

Interpretation...

When does the CIA constraint bind?

No arbitrage:

$$1 + i = (1 + r)(1 + \pi) = [f'(k) + 1 - \delta] p'/p = 1$$

The CIA constraint binds unless the return on money equals that on capital

• i.e. the nominal interest rate is zero.

Holding money has no opportunity cost.

The presence of money does not distort the intertemporal allocation.

We have the standard Euler equation.

Sequences $\{c_t, m_t, k_t\}$ that satisfy

- 1. Euler equation
- 2. budget constraint
- 3. no arbitrage

Plus boundary conditions

3.3. Binding CIA constraint

Euler equation:

$$u'(c) = \beta^2 u'(c'')(p'/p'')f'(k') + (1-\delta)\beta u'(c')$$
(5)

Today:

• Give up
$$dc = -\varepsilon$$
.

Tomorrow:

 $\blacktriangleright dk' = \varepsilon.$

• Eat the undepreciated capital: $dc' = (1 - \delta)\varepsilon$.

• Produce additional output $f'(k')\varepsilon$.

Save it as money: $dm'' = f'(k')\varepsilon p'$.

The day after:



Household Problem

Why isn't there a simple Euler equation for the perturbation:

- 1. $dc = -\varepsilon$. $dm' = p\varepsilon$.
- 2. $dc' = \varepsilon p/p'$.

The Euler equation for this perturbation is:

$$u'(c) = \lambda + \gamma$$

= $\beta u'(c') p/p' + \gamma$

Sequences $\{c_t, m_t, k_t\}$ that satisfy:

- 1. Euler equation
- 2. budget constraint
- 3. CIA constraint

Plus boundary conditions

4. Equilibrium

Government

The government's only role is to hand out lump-sum transfers of money.

The nominal money growth rule is

 $\tau_t = g \times M_{t-1}$

g > 0 is a parameter

Money holdings in period t are

$$M_t = M_{t-1} + \tau_t$$
$$= (1+g)M_{t-1}$$

Market clearing

- Goods: $c + k' = f(k) + (1 \delta)k$.
- Money market: $M_t = m_t$

Equilibrium

An **equilibrium** is a sequence that satisfies

5. Steady State

Steady state properties

Objects: $c, k, m/p, \pi$ Constant m/p requires

$$\frac{m_{t+1}}{m_t} = 1 + g = \frac{p_{t+1}}{p_t} = 1 + \pi \tag{6}$$

Steady State: CIA does not bind

$$f'(k) + 1 - \delta = (1 + g)^{-1} \text{ (no arbitrage)}$$
(7)
$$= 1/\beta \text{ (Euler)}$$
(8)
$$f(k) - \delta k = c \text{ (R.C.)}$$
(9)

Result:

A steady state where CIA does not bind only exists if $\beta = 1 + g$.

Intuition...

Then: The steady state coincides with the (Pareto optimal) non-monetary economy.

Binding CIA constraint

The Euler equation implies

$$1 = \beta^2 (1+\pi)^{-1} f'(k') + (1-\delta)\beta$$

Using $1 + \pi = 1 + g$ this can be solved for the capital stock:

$$f'(k_{ss}) = (1+g)[1-\beta(1-\delta)]/\beta^2$$
(10)

When $\beta = 1 + g$, this coincides with k_{ss} when the CIA constraint does not bind.

When does CIA constraint bind?

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Steady state return on money: (1+g)^{-1}
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If $(1+g) = \beta$:

- return on money equals return on capital (equals discount factor)
- CIA does not bind

Higher g reduces k_{ss} and increases return on capital Therefore: CIA binds when $(1+g) > \beta$

CIA implies:

$$f(k) = m/p \tag{11}$$

Goods market clearing with constant k implies

$$c = f(k) - \delta k \tag{12}$$

A steady state is a vector (c,k,m/p) that satisfies (10) through (12).

Definition

Money is called **neutral** if changing the level of M does not affect the real allocation.

It is called **super neutral** if changing the growth rate of M does not affect the real allocation.

Money is not super neutral

- Higher inflation (g) implies a lower k.
- Inflation increases the cost of holding money, which is required for investment (inflation tax).

Exercise:

- Show that super-neutrality would be restored, if the CIA constraint applied only to consumption $(m/p \ge c)$.
- Intuition: investment is no longer subject to the inflation tax.

The velocity of money is one

- Higher inflation reduces money demand only be reducing output.
- This is a direct consequence of the rigid CIA constraint and probably an undesirable result.
- Obviously, this would not be a good model of hyperinflation.
- This limitation can be avoided by changing the transactions technology (see RQ).

What if
$$(1+g) < \beta$$

There is no steady state with $1 + g < \beta$

The reason:

- money would offer a rate of return above the discount rate
- the household would choose unbounded consumption.
- Cf. the Euler equation

$$u'(c) = \beta R \ u'(c') \tag{13}$$

with $R = (1+g)^{-1}$ for holding money.

What would the equilibrium look like?

Optimal Monetary Policy

► The Friedman rule maximizes steady state welfare.

- Friedman Rule: Set nominal interest rate to 0.
- Proof: Under the Friedman rule, the steady state conditions of the CE coincides with the non-monetary economy's.

Intuition:

- It is optimal to make holding money costless b/c money can be costlessly produced.
- This requires that the rate of return on money $\frac{1}{1+\pi}$ equal that on capital.

Recall the central questions of monetary theory:

- 1. Why do people hold money, an asset that does not pay interest (rate of return dominance)?
- 2. Why is money valued in equilibrium?
- 3. What are the effects of monetary policy: one time increases in the money supply or changes in the money growth rate?

Is this a good theory of money?

Positive features:

- 1. Rate of return dominance.
- 2. Money plays a liquidity role.

Drawbacks:

- 1. The reason why money is needed for transactions is not modeled.
- 2. The form of the CIA constraint is arbitrary (and important for the results).
- 3. The velocity of money is fixed.



