AK Model: Phase Diagram

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Introduction

We study an endogenous growth model with transitional dynamics.

The model is asymptotically AK.

As an example of a phase diagram with endogenous growth.

The Model

We modify the Ak model's production function:

$$H(K,L) = AK + F(K,L) \tag{1}$$

In intensive form

$$h(k) = Ak + f(k)$$

where $F(K,L) = Lf(k)$ satisfies Inada conditions and has constant returns to scale in K and L jointly.

For simplicity, assume $f(k) = k^{\alpha}$ with $\alpha < 1$.

Equilibrium

The only change to the equilibrium conditions of the Ak model: the marginal product of capital is not A but

$$H_K(K,L) = A + F_K(K,L) = A + f'(k) \tag{2}$$

Laws of motion:

$$\begin{aligned} \dot{k} &= h(k) - (n+\delta)k - c && (3) \\ g(c) &= (h'(k) - \delta - \rho)/\sigma && (4) \end{aligned}$$

Asymptotically, $f'(k) \to 0$ and the model becomes Ak.

Phase Diagram with Endogenous Growth

How to draw a phase diagram when c and k grow at endogenous rates?

One approach: Find ratios that are constant asymptotically

For inspiration, start from

$$g(k) = h(k)/k - (n+\delta) - c/k \tag{5}$$

That suggests to try:

- $z = h(k)/k$
- $x = c/k$.

Another approach: **detrend** the model and then draw the phase diagram.

Laws of motion

$$
\begin{aligned}
g(z) &= g(h(k)) - g(k) \qquad (6) \\
g(x) &= g(c) - g(k) \qquad (7)
\end{aligned}
$$

- We therefore need to find expressions for $g(h(k))$, $g(k)$, and $g(c)$ in terms of z and x only.
- First rewrite the law of motion for k as

$$
\begin{aligned}
g(k) &= h(k)/k - \delta - n - c/k \qquad (8) \\
&= z - \delta - n - x \qquad (9)
\end{aligned}
$$

Laws of motion

- Next, $g(c) = [h'(k) - \delta - \rho]/\sigma$.
  - We need to replace $h'(k)$.
- Note that

$$h'(k) = A + \alpha f(k)/k = A + \alpha(z - A) = \alpha z + (1 - \alpha) A$$

- Use this to rewrite (4) as

$$g(c) = \frac{\alpha z + (1 - \alpha) A - \delta - \rho}{\sigma}$$

Laws of motion

Finally,

$$g(h(k)) = \frac{h'(k)k}{h(k)} g(k) = \frac{\alpha z + (1-\alpha)A}{z} g(k)$$

Laws of motion

$$
\begin{aligned}
g(z) &= g(h(k)) - g(k) \\
&= [\frac{\alpha z + (1-\alpha)A}{z} - 1][z - x - n - \delta] \\
&= (1-\alpha)(A/z - 1)[z - x - n - \delta]
\end{aligned}
$$

and

$$
\begin{aligned}
g(x) &= g(c) - g(k) \\
&= \frac{\alpha z + (1-\alpha)A - \rho - \delta}{\sigma} - z + x + n + \delta \\
&= \varphi + x + z(\alpha/\sigma - 1)
\end{aligned}
$$

where $\varphi = n + \delta + (1-\alpha)A/\sigma - (\rho+\delta)/\sigma$.

Phase diagram

$\dot{x} = 0$ requires

$$x_{ss} = (1 - \alpha/\sigma) z_{ss} - \varphi \tag{10}$$

For realistic parameter values (e.g. $\alpha \simeq 0.3$ and $\sigma \geq 1$), we have $0 < 1 - \alpha/\sigma < 1$.

- Negative intercept.
- Slope < 1.

$\dot{z} = 0$ Locus

$\dot{z} = 0$ has two solutions:

- $z = A$ or
- $x = z - n - \delta$.

In steady state:

$$z_{ss} = A \tag{11}$$

because

$$\lim_{k\to\infty} z = \lim_{k\to\infty} \frac{Ak + f(k)}{k} = A \tag{12}$$

$\dot{z} = 0$ Locus

But for finite k: $z = A/k + f(k)/k > A$.

Therefore the relevant condition is

$$x = z - n - \delta \tag{13}$$

- Negative intercept
- Slope $= 1$

Summary

Laws of motion:

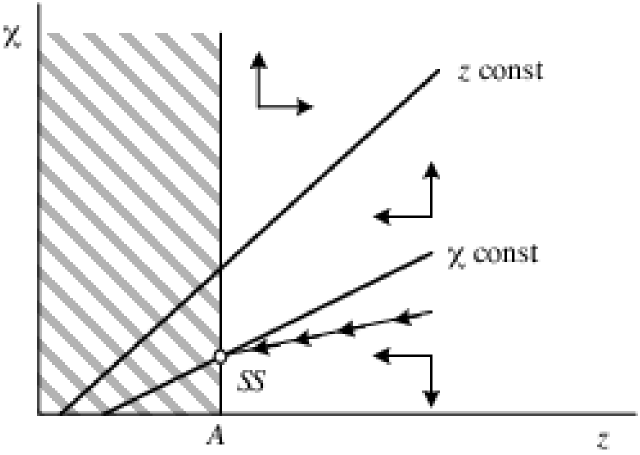
$$\begin{aligned}\dot{z} &= (1-\alpha)(A-z)(z-x-n-\delta)\\ g(x) &= \varphi + x + z(\alpha/\sigma - 1)\end{aligned}$$

$$\begin{aligned}\dot{z} &= 0 : x = z - n - \delta\\ \dot{x} &= 0 : x = -\varphi + (1-\alpha/\sigma)z\end{aligned}$$

Steady state: $z = A$.

Otherwise: $z > A$

Phase Diagram with Endogenous Growth



Phase Diagram with Endogenous Growth

This system is **saddle-path stable**.

Consider x_0 above the saddle path

- $x = c/k$ grows over time; in fact $g(x) > 0$ (strictly above $\dot{x} = 0$) implies $x \to \infty$
- $z = h(k)/k$ grows over time; that means that k must shrink
- feasibility requires $g(k) = h(k)/k - \delta - c/k$; therefore $k \to 0$
- but then $g(c) \to \infty$ which violates TVC and feasibility

Stability

Consider x_0 below the saddle path

- if above $\dot{x} = 0$: it eventually crosses above the saddle, which is impossible
- otherwise $g(z) < 0$ until $z \to A$ which implies $k \to \infty$
- then $g(u') \to A - \delta - \rho$
- this violates TVC: $\lim_{t\to\infty} e^{-\rho t} u'(c_t) k_t \to \infty$

Both x and z converge monotonically to the steady state.

Summary

The important point is the general approach for dealing with the dynamics of growing economies:

1. Write out the equilibrium conditions as usual.
2. Find conditions characterizing the balanced growth path.
3. Find ratios that are constant on the balanced growth path (x and z).
4. Express the laws of motion of the economy in terms of these ratios.

An alternative approach is to transform the economy into stationary form before characterizing its equilibrium.

Reading

- Acemoglu (2009), ch. 11.
- Krusell (2014), ch. 8.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.