Endogenous Growth: AK Model

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Endogenous Growth

- Why do countries grow?
  - A question with large welfare consequences.
- We need models where growth is endogenous.
- The simplest model is a variation of the Ramsey model.
  - Growth can be sustained if the MPK is bounded below.
  - AK model

Necessary Conditions for Sustained Growth

- How can growth be sustained without exogenous productivity growth?
- A necessary condition: **constant returns to the reproducible factors**.
  - The production functions for inputs that can be accumulated must be linear in those inputs.
  - Example: In the growth model, K would have to be produced with a technology that is linear in K
- This motivates a simple class of models in which
  1. only K can be produced and
  2. the production function is AK.
- This can be thought of as a reduced form for more complex models (we'll see examples).

Solow AK model

To see what is required for endogenous growth, consider the Solow model:

$$g(k) = sf(k)/k - (n+\delta) \tag{1}$$

Positive long-run growth requires: As $k \to \infty$ it is the case that

$$f(k)/k > n+\delta \tag{2}$$

L'Hopital's rule implies (if f' has a limit):

$$\lim f(k)/k = \lim f'(k) \tag{3}$$

Sustained growth therefore requires:

$$\lim_{k\to\infty} f'(k) > n+\delta \tag{4}$$

Necessary Conditions for Sustained Growth

- This argument is more general than the Solow model.
  - It does not matter how s is determined.
- If $\lim_{k\to\infty} f'(k)$ exists, the production function has **asymptotic constant returns to scale**.

$$f(k) \to Ak + B \tag{5}$$

- It is fine to have diminishing returns for finite k.

Examples

1. $f(k) = Ak + Bk^{\alpha}$ with $0 < \alpha < 1$
   1.1 $f(k)/k \to A$
2. CES production function with high elasticity of substitution:

$$F(K,L) = \left[\mu K^{\theta} + (1-\mu) L^{\theta}\right]^{1/\theta} \tag{6}$$

   2.1 $f(k) = \left[\mu k^{\theta} + 1 - \mu\right]^{1/\theta}$
   2.2 Elasticity of substitution: $\varepsilon = (1-\theta)^{-1}$.
   2.3 If $\theta > 0$ $[\varepsilon > 1]$, $f(k)/k \to \mu^{1/\theta}$.

AK Solow Model

- In the Solow model, assume $f(k) = Ak$.
- Law of motion:

$$g(k) = sA - n - \delta \tag{7}$$

- Changes in parameters alter the growth rate of k.
- The model does not have any transitional dynamics: k always grows at rate $sA - n - \delta$.

AK Solow Model

- It is not necessary to have constant returns in all sectors of the economy.
- Imagine that c is produced from k with diminishing returns to scale: $c = [(1-s)Ak]^{\varphi}$ with $\varphi < 1$.
- The law of motion for k is unchanged (so is the balanced growth rate of k).
- This model still has a balanced growth path with a strictly positive growth rate, but not c and k grow at different constant rates:

$$g(c) = \varphi\, g(k) \tag{8}$$

AK Neoclassical Growth Model

AK neoclassical growth model

This model adds optimizing consumers to the Ak model.

Households maximize

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t)\, dt \tag{9}$$

subject to the flow budget constraint

$$\dot{k} = (r-n)k - c \tag{10}$$

There is no labor income because in the Ak world all income goes to capital.

AK neoclassical growth model

For balanced growth we need

$$u(c) = c^{1-\sigma}/(1-\sigma) \tag{11}$$

The optimality conditions are the same as in the Cass-Koopmans model:

$$g(c) = (r-\rho)/\sigma$$

and the transversality condition (assuming constant r)

$$\lim_{t\to\infty} k_t e^{-(r-n)t} = 0 \tag{12}$$

Firms

Firms maximize period profits.

The first-order condition is $r = A - \delta$.

Equilibrium

An allocation: $c(t), k(t)$.

A price system: $r(t)$.

These satisfy:

1. Household: Euler, budget constraint (TVC).
2. Firm: 1 foc.
3. Market clearing:

$$\dot{k} = Ak - (n+\delta)k - c \tag{13}$$

Summary

Simplify into a pair of differential equations:

$$\begin{aligned}\dot{k} &= (A-\delta-n)k-c \qquad (14)\\ g(c) &= (A-\delta-\rho)/\sigma \qquad (15)\end{aligned}$$

Boundary conditions: k_0 given and the TVC.

Of course, we could have simply taken the equilibrium of the standard growth model and replaced $f'(k) = A$ and $f(k) = Ak$.

Bounded utility

We need restrictions on the parameters that ensure bounded utility.

Lifetime utility is

$$\int_0^\infty e^{-(\rho-n)t}\left[c_0\, e^{g(c)t}\right]^{1-\sigma} dt/(1-\sigma) \tag{16}$$

Boundedness then requires that $n-\rho+(1-\sigma)g(c)<0$.

Instantaneous utility cannot grow faster than the discount factor $(\rho-n)$.

Transitional dynamics

This model has no transitional dynamics.

Consumption growth is obviously constant over time.

To show that $g(k)$ is constant: we need to solve for $k(t)$ in closed form.

Details

Summary

The AK model has a very simple equilibrium.

1. The saving rate is constant.
2. All growth rates are constant.

This is very convenient, but also very limiting in many applications.

How to think about AK models?

In the data, there is at least one non-reproducible factor: labor.

Do models with constant returns to reproducible factors make sense?

The best way of thinking about AK models:

- a **reduced form** for a model with multiple factors
- there may be transition dynamics, but it does not matter if you are interested in long-run issues
- there may be fixed factors, but it does not matter if there are constant returns to reproducible factors.

Examples: AK as reduced form

1. Human capital: $F(K, hL)$ with K and h reproducible.
2. Externalities:
   2.1 Romer (1986). For the firm $F(k_i, l_i K) = K^{1-\alpha} k_i^{\alpha} l_i^{\theta}$
   2.2 Firms take K as given - diminishing returns to k_i.
   2.3 In equilibrium: $K = \sum k_i$ - constant returns to scale to K.
3. Increasing returns to scale at the firm level: $y = Ak^{\alpha} l^{1-\alpha}$
   3.1 A can be produced somehow - R&D.
   3.2 Need imperfect competition.

Example: Lucas (1988)

Example: Lucas (1988)

A classic endogenous growth paper.

Growth is due to human capital accumulation.

The model has an AK reduced form.

Model: Lucas (1988)

Demographics:

- A representative, infinitely lived household.

Preferences:

$$\int_0^\infty e^{-\rho t} u(c_t)\, dt \tag{17}$$

$$u(c) = c^{1-\sigma}/(1-\sigma) \tag{18}$$

Technology:

$$\dot{k} + c = f(k,h,l) - \delta k \tag{19}$$

$$\dot{h} = e(k,h,l) - \delta h \tag{20}$$

where $f(k,h,l) = k^\alpha (lh)^{1-\alpha}$ and $e(k,h,l) = B(1-l)h$

Lucas (1988): Balanced growth rates

Law of motion for h:

$$g(h) = B(1-l) - \delta \tag{21}$$

Law of motion for k:

$$g(k) + c/k = (lh/k)^{1-\alpha} - \delta \tag{22}$$

Therefore:

$$g(c) = g(k) = g(h) \tag{23}$$

Lucas (1988): Optimality

Current value Hamiltonian:

$$H = u(c) + \lambda \left[e(k,h,l) - \delta h\right] + \mu \left[f(k,h,l) - \delta k - c\right] \tag{24}$$

FOCs:

$$\partial H/\partial c = u'(c) - \mu = 0 \tag{25}$$

$$\partial H/\partial u = \lambda e_l + \mu f_l = 0 \tag{26}$$

$$\rho \lambda - \dot{\lambda} = \lambda \left[e_h - \delta\right] + \mu f_h \tag{27}$$

$$\rho \mu - \dot{\mu} = \mu \left[f_k - \delta\right] + \lambda e_k \tag{28}$$

Major simplification from $e_k = 0$.

Optimality

Euler equation (using $e_k = 0$)

$$g(c) = \frac{f_k - \delta - \rho}{\sigma} \tag{29}$$

From FOC for h:

$$-g(\lambda) = e_h - \delta - \rho + \mu/\lambda f_h \tag{30}$$

FOC for u:

$$\frac{\mu}{\lambda} f_h = -\frac{e_l f_h}{f_u} = (Bh)\frac{l}{h} = Bl \tag{31}$$

Substitute into FOC for h:

$$-g(\lambda) = B - \delta - \rho \tag{32}$$

This is an exogenous constant!

Balanced growth

Constant f_k requires constant k/h.

Then

$$\frac{\mu}{\lambda} = \frac{Bh}{f_l} = \frac{Bh}{(1-\alpha)(k/h)^{\alpha} l^{1-\alpha}} \tag{33}$$

requires constant μ/λ.

Then $g(u_c) = -g(\mu) = -g(\lambda) = B - \delta - \rho$.

This determines the interest rate:

$$r = f_k - \delta = B - \delta \tag{34}$$

The balanced growth rate is determined by the linear human capital technology:

$$g(c) = \frac{B - \delta - \rho}{\sigma} \tag{35}$$

Intuition

- The household has 2 assets: k and h.
- One asset has a constant rate of return:
  - give up 1 unit of time to gain a fixed increment of future income
  - regardless of current values of k and h.
- This pins down the interest rate on the other asset by no arbitrage.
- All of this has implicitly assumed an interior solution!

Summary

Sustained growth requires that inputs are produced with constant returns to reproducible inputs.

Then the model is (at least asymptotically) of the AK form: $\dot{K} = AK$.

The AK model is a reduced form of something more interesting.

Reading

- Acemoglu (2009), ch. 11.
- Krueger, "Macroeconomic Theory," ch. 9.
- Krusell (2014), ch. 8.
- Barro and Martin (1995), ch. 1.3, 4.1, 4.2, 4.5.
- Jones and Manuelli (1990)
- Lucas (1988).

Digression: Solving for k(t) I

- Law of motion:

$$\dot{k}_t = (A - \delta - n) k_t - c_0 \ \exp\left(\frac{A - \delta - \rho}{\sigma} t\right) \tag{36}$$

- Solution to $\dot{x} = ax - b(t)$ is

$$x_t = x_0 e^{at} - e^{at} \int_0^t e^{-as} b(s) ds \tag{37}$$

- To verify:

$$\begin{aligned} \dot{x}_t &= a x_0 e^{at} - a e^{at} \int_0^t e^{-as} b(s) ds - e^{at} e^{-at} b(t) && (38) \\ &= a x_t - b(t) && (39) \end{aligned}$$

Digression: Solving for k(t) II

- Define

$$a = r - n = A - \delta - n > 0 \tag{40}$$

$$b = g_c = \frac{A - \delta - \rho}{\sigma} > 0 \tag{41}$$

- Then

$$k_t = k_0 \exp(at) - \exp(at) \int_0^t c_0 \exp([-a+b]s)\, ds \tag{42}$$

- Note:

$$\int_0^t e^{zs} ds = \frac{e^{zt} - 1}{z} \tag{43}$$

Digression: Solving for k(t) III

- Therefore:

$$k_t = k_0 e^{at} - \frac{c_0}{b-a} e^{at} \left[e^{(b-a)t} - 1 \right] \tag{44}$$

$$= \left[k_0 + \frac{c_0}{b-a} \right] e^{at} - \frac{c_0}{b-a} e^{bt} \tag{45}$$

- Now we show that $g(k)$ is constant: $k_t = k_0 e^{bt}$.
- Transversality:

$$\lim_{t-\infty} e^{(r-n)t} k_t = 0 \tag{46}$$

- Note that $a = r - n = A - \rho - n$.
- If $b > a$: $g(k) \to b > a$ and TVC is violated.
- So we need $b < a$.

Digression: Solving for k(t) IV

- With $b < a$ capital grows at rate a, unless the term in brackets is 0:

$$k_0 + \frac{c_0}{b-a} = 0 \tag{47}$$

- If $g(k) = a$, then $g\left(e^{-(r-n)t}k_t\right) = 0$ - because $a = r - n$.
  - That violates TVC.
- The only value of c_0 consistent with TVC is the one that sets the term in brackets to 0.
- It implies that k always grows at rate b.

Saving rate

- We can solve for c/k and the saving rate.

$$\begin{aligned} g(k) - g(c) &= [A - \delta - n - c/k] - (A - \delta - \rho)/\sigma = 0 \\ c/k &= A - \delta - n - (A - \delta - \rho)/\sigma \end{aligned}$$

- And the gross savings rate is

$$\begin{aligned} s &= (\dot{K} + \delta K)/AK \\ &= [g(K) + \delta]/A \\ &= [g(c) + n + \delta]/A \\ &= [(A - \delta - \rho)/\sigma + n + \delta]/A \end{aligned}$$

- The savings rate is high, if (σ, ρ or A) are low, or if n is high.

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