Midterm Exam. Econ720. Fall 2024

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- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 75 minutes.
- The total number of points is 75.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 OLG with Different Discount Rates

Demographics:

- Time is discrete and goes on forever.
- At each date t, $N_t = (1+n)^t$ young are born. Each lives for 2 periods.
- Fraction μ_j of each cohort are of type j. We have two types: patient and impatient.

Preferences: $\mathcal{U}(c_{j,t}) + \mathcal{V}(l_{j,t}) + \beta_j \mathcal{U}(z_{j,t+1})$ where $\mathcal{U}$ is well behaved.

- c is young consumption. z is old consumption. $0 \leq l \leq 1$ is leisure.
- Assume $\beta_1 > \beta_2$ so that type 1 households are patient relative to type 2 households.

Endowments:

- Each young has one unit of time which can be used for leisure l or work $1-l$.
- Each old household receives an endowment of e goods.

Technology: $Y_t = N_{t-1}e + \mathcal{F}(L_t) = C_t$ with $\mathcal{F}(L_t) = AL_t$ where $L_t = \sum_j \mu_j N_t (1 - l_{j,t})$ is aggregate labor supply and C is aggregate consumption.

Markets: There are competitive markets for goods (numeraire), labor (wage w), and one period bonds (gross real interest rate R). Bonds are issued and purchased by households (in zero net supply).

Consider an equilibrium in sequence language.

Questions:

1. [10 points] State the household problem for type j.

   Answer

   This is standard with budget constraints $w(1-l) = c + b'$ and $Rb + e = z$.

   The household solves

   $$\max_{l,b'} \mathcal{U}(w(1-l) - b') + \mathcal{V}(l) + \beta \mathcal{U}(R'b' + e)$$

   .

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2. [10 points] Derive an Euler equation and the static optimality condition $w\mathcal{U}'(c) = \mathcal{V}'(l)$. Explain what those say in words. Define a solution to the household problem.

   Answer

   FOCs are the static condition given in the question and the Euler equation $\mathcal{U}'(c) = \beta_j \mathcal{U}'(z') R'$. Interpretation is standard.

   A solution is (c, z', l, b') that satisfy 2 focs and 2 budget constraints.

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3. [10 points] Define a competitive equilibrium.

Answer

Equations:

- household: 4
- firm: $w = A$
- goods market (RC), labor market (given in the question text), bond market: $\sum_j \mu_j b'_j = 0$.

Objects: c, z, l, b, w, R, L.

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4. [10 points] Show that patient households save while impatient households borrow in equilibrium. From hereon assume CRRA preferences of the form $\mathcal{U}(x) = \mathcal{V}(x) = x^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$. Hint: Try two cases, one where the impatient have higher labor supply than the patient and another where it's the other way around.

Answer

The Euler equation is now $z'/c = (\beta_j R')^{\sigma}$. Patient households have faster consumption growth.

Try a case where the impatient household saves more than the patient household for given R'.

If the impatient household has higher labor supply, then (by the static condition) they also have lower c. From the Euler equation, they have lower z'. But that cannot be because they have higher lifetime incomes.

If the impatient household has lower labor supply, then they have higher c and therefore lower saving. Since aggregate savings equal zero, the impatient must borrow and the patient must save. So the case where the impatient save more cannot be an equilibrium.

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5. [10 points] Now assume that households cannot borrow ($b \geq 0$). Derive two equations that solve for equilibrium l and R. Hint: Whose Euler equation holds with equality?

Answer

For any given R', the patient household wants to save more than the impatient one (see answer above). One household must violate its Euler equation. That can only be the one who wants to borrow (the impatient household).

Hence the equilibrium interest rate must make the patient household indifferent between borrowing and lending. It is fully determined by the patient household's problem.

The household solution now boils down to:

(a) old budget constraint without saving: $z = e$

(b) young budget constraint without saving: $c = A(1-l)$

(c) Euler equation: $\frac{z'}{c} = \frac{e}{A(1-l)} = (\beta_j R')^{1/\sigma}$ for the patient household only.

(d) Static optimality: $Ac^{-\sigma} = l^{-\sigma}$ or $c = A^{1/\sigma} l$.

Putting all together yields the solution for the interest rate: the Euler equation with the solution to $A(1-l) = A^{1/\sigma} l$ substituted in.

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2 Planning Problem

Demographics:

- Time is discrete and goes on forever.
- At date t, we have N_j households of type $j \in \{1, ..., n\}$.
- All households of a type are identical and live forever.

Preferences:

- A type j household maximizes $\sum_{t=0}^{\infty} \beta^t \mathcal{U}_j(c_{j,t}, l_{j,t})$.
- Assume that $\mathcal{U}_j$ is well behaved for every type.

Technology:

- There is one good in each period with resource constraint $\mathcal{F}(K_t, L_t) = C_t + K_{t+1}$.
- $\mathcal{F}$ is well behaved. K is capital.
- $C_t = \sum_j N_j c_{j,t}$ is aggregate consumption. $L_t = \sum_j N_j (1 - l_{j,t})$ is aggregate labor input.

Endowments: At time 0, the planner has K_0 units of capital.

Questions:

1. [13 points] Write down the planner's problem in sequence language. Let μ_j denote how much the planner likes each individual of type j.

   Answer

   The planner maximizes welfare

$$\max_{\{c_{j,t}, l_{j,t}, K_t\}} \sum_{j=1}^{n} \mu_j N_j \sum_t \beta^t \mathcal{U}_j(c_{j,t}, l_{j,t})$$

   subject to resource constraints

$$\mathcal{F}\left(K_t, \sum_j N_j (1 - l_{j,t})\right) = \sum_j N_j c_{j,t} + K_{t+1}$$

   μ_j is the planner's weight for type j. This is expressed as how much the planner likes each individual of type j (as opposed to the entire cohort). That makes interpreting the first-order conditions easier.

2. [12 points] Write down the planner's problem as a Dynamic Program.

   Answer

The state variable is K.

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\begin{aligned}
V(K) = \max_{c_j, l_j, K'} & \sum_{j=1}^{n} \mu_j N_j \mathcal{U}_j (c_j, l_j) + \beta V(K') \\
& + \lambda \left[\mathcal{F}(K, L) - C - K' \right]
\end{aligned}
$$

where aggregates are defined above.

End of exam.