Midterm Exam. Econ720. Fall 2022

Professor Lutz Hendricks. UNC.

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 75 minutes.
- The total number of points is 75.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 OLG Model with T Period Lifetimes [41 points]

Demographics: Time is discrete and goes on forever. At each date t, N_t identical young are born. Each lives for T periods.

Preferences: A person born at τ values consumption according to

$$\sum_{a=1}^{T} \beta^a \mathcal{U}\left(c_{\tau,\tau+a-1}\right) \tag{1}$$

where $\mathcal{U}$ is well behaved. $c_{\tau,t}$ is consumption of a person born in τ at date t. Note that $t = \tau + a - 1$ for a person of age a.

Endowments: In period 1, agents are endowed with $k_{\tau,1}$ units of capital.

Technology: $Y_t = \mathcal{F}\left(K_t, L_t\right) = C_t + K_{t+1} - (1-\delta) K_t$ where C is aggregate consumption, K is aggregate capital, and L is aggregate labor input.

Markets: There are competitive markets for goods (numeraire), labor (wage w_t), and capital rental (price q_t). A representative firm operates the technology.

Consider an equilibrium in sequence language.

Questions:

1. [6 points] Write out the household problem as a Dynamic Program.

   Answer

   Bellman: $V(k,a) = \max_{k'} \mathcal{U}\left(w + [q + 1 - \delta] k - k'\right) + \beta V\left(k', a+1\right)$ with $V(k, T+1) = 0$.

   Note that there is a different Bellman equation for each age. This is why a is a state variable.

   The question did not clearly state that households work one unit of time in each period.

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2. [6 points] Derive the Euler equation and define a solution to the household problem in sequence language.

   Answer

   Euler equations are standard for interior periods. In the terminal period, of course $k' = 0$.

   Solution: $c_{\tau,t}$, $k_{\tau,t}$ that solve Euler equations, $k_{\tau,T+1} = 0$ and initial condition $k_{\tau,1}$ given. There is no TVC.

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3. [6 points] Write out the market clearing conditions.

   Answer

Aggregation function: $\mathcal{A}(x,t) = \sum_{\tau=t-T+1}^{t} N_\tau x_{\tau,t}$. Each aggregate sums over all of the cohorts that are alive at each date.

Capital and labor: $K_t = \mathcal{A}(k,t)$ and $L_t = \mathcal{A}(1,t) = \sum_{\tau=t-T+1}^{t} N_\tau$.

Goods: RC with $C_t = \mathcal{A}(c,t)$

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4. [6 points] Define a competitive equilibrium in sequence language.

Answer

Objects: $\{c_{\tau,t}, k_{\tau,t}, K_t, L_t, C_t\}$ and prices $\{w_t, q_t\}$

Conditions: Household $(2 \times T)$, firms (standard, 2), market clearing (3).

Now consider the Recursive Competitive Equilibrium. For simplicity, assume that the mass of households aged a equals μ_a (because working with population growth requires detrending, which is a bit complicated for a midterm question).

5. [4 points] What is the aggregate state; and why?

Answer

Aggregate state S: capital holdings of all agents; a vector with elements κ_a. Generically, the aggregate state contains the joint distribution of individual states. In this case, that's κ_a because age is the only source of heterogeneity.

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6. [6 points] Define a solution to the household problem in recursive language (no need to rederive the household's first-order conditions).

Answer

- Objects: Policy functions for consumption $\mathcal{C}(k,a,S)$ and saving $\mathcal{K}(k,a,S)$ and value functions $V(k,a,S)$.
- Conditions:
  - Given $V(k,a+1,S')$, the policy functions solve the RHS of the age a Bellman equation. Note that we want to think for this V as a function of k (or k'; that's just notation) for given $a+1$ and S'.
  - $V(k,a,S) = \mathcal{U}(\mathcal{W}(S) + \mathcal{Q}(S)k + (1-\delta)k - \mathcal{K}(k,a,S)) + \beta V(\mathcal{K}(k,a,S), a+1, \mathcal{G}(S))$.
  - This takes the place of the fixed point property in the infinite horizon problems that we often study.

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7. [7 points] State the market clearing conditions.

Answer

Market clearing:

- Goods: $\mathcal{F}(K(S), L(S)) + (1-\delta) K(S) = C(S) + K'(S)$
- Aggregation function: $\mathcal{A}(x, S) = \sum_{a=1}^{T} \mu_a x(\kappa_a(S), a, S)$.
- Then $K'(S) = \mathcal{A}(\mathcal{K}, S)$ where (remember) $\mathcal{K}(k, a, S)$ is the decision rule for k'. Other aggregates are similar.

Everything should be a function of the aggregate state.

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8. [6 points] Define a Recursive Competitive Equilibrium.

Answer

Objects:

- Household: Policy functions for consumption $\mathcal{C}(k, a, S)$ and saving $\mathcal{K}(k, a, S)$ and value functions $V(k, a, S)$.
- Price functions: $\mathcal{W}(S)$ and $Q(S)$
- Law of motion for the aggregate state $S' = \mathcal{G}(S)$.

Conditions:

- Household: above
- Firms: standard. Factor prices equal marginal products.
- Market clearing: above.
- Consistency: $\mathcal{K}(\kappa_a, a, S) = \mathcal{G}_a(S)$ where $\mathcal{G}_a$ is the a-th row of $\mathcal{G}$. In words: The household's choice of k' (given that they hold the equilibrium $k = \kappa_a(S)$) must be consistent with the "expected" $\kappa_a(S')$.

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2 Bonds Of Different Maturities [34 points]

This question examines Ricardian Equivalence when the government issues bonds of different maturities to finance spending.

Demographics: There is a representative agent of unit mass who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$

Technology: $\mathcal{F}(K, L) + (1 - \delta) K = C + G + K'$. $\mathcal{F}$ has constant returns to scale.

Endowments:

- At the beginning of time, the household is endowed with k_0 units of capital, b_0 one period bonds, and B_0 forever bonds.
- Each forever bond pays one unit of consumption forever. Its price is p_t.
- Each one period bond has price 1 and pays R_{t+1} units of consumption next period.

Government: The government imposes a lump-sum tax τ_t and spends G_t units of the good on a useless purpose.

Firms are standard with first-order conditions $q = f'(k)$ and $w = f(k) - f'(k) k$ where $f(k) = \mathcal{F}(K/L, 1)$.

Questions:

1. [4 points] State the household's Dynamic Program, where the budget constraint is given by

$$k_{t+1} + c_t + \tau_t + b_{t+1} + p_t (B_{t+1} - B_t) = (q_t + 1 - \delta) k_t + w_t + R_t b_t + B_t$$

Answer

$V(k, b, B) = \max\ u([q + 1 - \delta] k + w - \tau + Rb + B - k' - b' - p(B' - B)) + \beta V(k', b', B')$

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2. [10 points] Derive the household's first-order conditions and define a solution in sequence language. Interpret the first-order conditions.

Answer

The first-order conditions may be written as

$$\begin{aligned} u'(c) &= \beta R' u'(c') \\ R &= q + 1 - \delta \\ R' &= (1 + p')/p \end{aligned}$$

A solution consists of sequences (c, k, b, B) that satisfy the 3 FOCs and the budget constraints. Boundary conditions are: k_0, b_0, B_0 given and the transversality condition

$$\lim_{t \to \infty} \beta^t u'(c_t) [k_t + b_t + p_t B_t] = 0$$

Of course, what is really going on here is the portfolio composition is indeterminate. This shows up as two no-arbitrage type first-order conditions.

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3. [8 points] The government flow budget constraint is given by

$$G_t + R_t\, b_t + B_t = \tau_t + b_{t+1} + p_t (B_{t+1} - B_t)$$

Show that the present value budget constraint of the government can be written as

$$b_0 + (1 + p_0)\, B_0 / R_0 = \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{D_t}$$

where $D_t = R_0 \cdot ... \cdot R_t$ is a cumulative discount factor.

Answer

This is (mostly) a standard forward replacement argument. Start from $b_{t+1} = R_t b_t - S_t$ where $S_t \equiv \tau_t - G_t + p_t\, B_{t+1} - (1 + p_t)\, B_t$ is the government's "saving" (sort of).

For $t = 0$:

$$\begin{aligned} b_0 &= R_0^{-1}\, [S_0 + b_1] \\ &= R_0^{-1} S_0 + (R_0 R_1)^{-1}\, [S_1 + b_2] \end{aligned}$$

Iterating over this implies

$$\begin{aligned} b_0 &= \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{D_t} + \sum_{t=0}^{\infty} \frac{p_t\, B_{t+1} - (1 + p_t)\, B_t}{D_t} \\ &= \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{D_t} + \sum_{t=1}^{\infty} \left[\frac{p_{t-1}\, B_t}{D_{t-1}} - \frac{(1 + p_t)\, B_t}{D_t} \right] - (1 + p_0)\, B_0 / R_0 \end{aligned}$$

Note that the lower bound of the sum has been changed to 1 and the only B_0 term has been pulled out of the sum. Next, I show that the term in the square brackets equals zero for each t. To see this, note that

$$\frac{p_{t-1}\, B_t}{D_{t-1}} = \frac{p_{t-1} R_t\, B_t}{D_t} = \frac{(1 + p_t)\, B_t}{D_t}$$

where the last equality holds because all assets pay the same rate of return. The intuition why the term in square brackets is zero is that future bond issues change the timing of government surpluses, but it does not add to the present value of resources the government can spend.

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4. [12 points] Show that Ricardian Equivalence holds in this economy. That is, a change in the timing of taxation does not affect the equilibrium allocation (for a given sequence G_t). The best way of answering this part is to define a competitive equilibrium in such a way that

a subset of equations which does not contain τ_t determines the allocation. Note that the household's present value budget constraint is given by

$$b_0 + \frac{(1+p_0)\,B_0}{R_0} + k_0 = \sum_{t=0}^{\infty} \frac{c_t + \tau_t - w_t}{D_t}$$

Answer

Substitute the government present value budget constraint into the household present value budget constraint to obtain

$$k_0 = \sum_{t=0}^{\infty} \frac{c_t + G_t - w_t}{D_t}$$

Define an equilibrium as sequences $\{c_t, k_t, K_t, L_t, \tau_t, b_t, B_t, q_t, w_t, p_t, R_t\}$ that satisfy:

- Household: $\{c_t, k_t\}$ solve the Euler equation $u'(c_t) = (q_{t+1} + 1 - \delta)\,\beta\,u'(c_{t+1})$ and the present value budget constraint. Also no-arbitrage conditions that ensure all assets pay the same rate of return.
- Firms: $\{q_t, w_t\}$ solve the 2 first-order conditions
- Government: $\{b_t, B_t, \tau_t\}$ solve the government present value budget constraint and the flow budget constraint (some indeterminacy remains).
- The goods market clears: $\mathcal{F}(K_t, 1) + (1-\delta)K_t = K_{t+1} + C_t + G_t$.
- Other market clearing: $K = k$ and $L = 1$.

Now the system of equilibrium conditions is block-recursive. The household, firm and market clearing conditions determine $\{c_t, k_t, q_t, w_t, p_t\}$; they do not depend on tax rates. The remaining variables, $\{\tau_t, b_t, B_t\}$, are determined by the government budget constraint. The timing of taxes is not determined.

For completeness, here is the derivation of the household's present value budget constraint. The household flow budget constraint is given by

$$b_t = R_t^{-1}\left[c_t + \tau_t - w_t + k_{t+1} - R_t k_t + p_t\,B_{t+1} - (1+p_t)\,B_t + b_{t+1}\right]$$

Iterating over this expression yields

$$\begin{aligned} b_0 &= \sum_{t=0}^{\infty} D_t^{-1}\left[c_t + \tau_t - w_t + k_{t+1} - R_t k_t + p_t\,B_{t+1} - (1+p_t)\,B_t\right] \\ &= \sum_{t=0}^{\infty} D_t^{-1}\left[c_t + \tau_t - w_t + k_{t+1} - k_{t+1} + p_t\,B_{t+1} - R_{t+1}\,(1+p_{t+1})\,B_{t+1}\right] - \frac{(1+p_0)\,B_0}{R_0} + k_0 \end{aligned}$$

The second equation is obtained by pulling the date 0 terms out of the sum. By the same argument as for the government we find that $p_t B_{t+1} - R_{t+1}\,(1+p_{t+1})\,B_{t+1} = 0$ and the asserted budget constraint follows. The logic is again that the present value of future dissaving must equal current wealth.

End of exam.