Midterm Exam. Econ720. Fall 2021

Professor Lutz Hendricks. UNC.

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 75 minutes.
- The total number of points is 75.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

---

1 OLG Model with Public Capital

Demographics: At each date t, mass 1 of identical young are born. Each lives for two periods.

Preferences: $\mathcal{U}\left(c_t^y\right) + \beta\mathcal{U}\left(c_{t+1}^o\right) - \mathcal{V}\left(n_t\right)$ where n denotes hours worked by the young. $\mathcal{V}$ is strictly increasing and strictly convex.

Technology: $y_t = A\left(\bar{k}\right) n_t$ where $\bar{k}$ is public capital. Resource constraint: $y_t = c_t^y + c_t^o + \delta\bar{k}$.

Government: The government prints money and uses the seignorage to buy $\delta\bar{k}$ units of the good (to keep the capital stock constant). $\delta\bar{k}p_t = M_{t+1} - M_t$.

Markets: There are competitive markets for consumption goods (price p_t), labor (wage w_t), and money (numeraire).

Questions:

1. [10 points] Write out a solution to the household problem.

   Answer ____________________________________

   The household solves the standard one-asset consumption savings problem, except that labor supply is endogenous.[1] Let $R_{t+1} = p_t/p_{t+1}$ denote the return on money.

   $$\max_{n,s} \mathcal{U}\left(w_t n_t - s_t\right) + \beta\mathcal{U}\left(R_{t+1}s_t\right) - \mathcal{V}\left(n_t\right) \tag{1}$$

   Then we have a standard Euler equation $\mathcal{U}'\left(c_t^y\right) = R_{t+1}\beta\mathcal{U}'\left(c_{t+1}^o\right)$ and static condition $\mathcal{U}'\left(c_t^y\right) w = \mathcal{V}'\left(n_t\right)$. A solution consists of $c_t^y, c_{t+1}^o, s_t, n_t$ that satisfy the 2 focs and 2 budget constraints.

   ---

2. [15 points] Define a competitive equilibrium.

   Answer ____________________________________

   Equilibrium objects: $c_t^y, c_t^o, s_t, n_t, m_t, \pi_t, R_t, w_t$

   Equations:

   - household: 4
   - government: $\delta\bar{k} = m_{t+1}\left(1 + \pi_{t+1}\right) - m_t$ where $m = M/P$.
   - firms: $w = A\left(\bar{k}\right)$
   - goods market: RC
   - money market: $s_t = m_{t+1}\left(1 + \pi_{t+1}\right)$; because young saving $p_t s_t = M_{t+1}$. The young in period t purchase the entire money stock, including the money issued in period t to pay for current capital investment.

[1] Based on a Penn State qualifying exam in 2016.

- identity: $R_{t+1} = (1 + \pi_{t+1})^{-1}$.

---

3. [10 points] Show that equilibrium boils down to $\{n_t, m_t, \pi_t\}_{\forall t}$ that satisfy

$$\mathcal{U}'\left(\bar{A}n_t - m_{t+1}(1 + \pi_{t+1})\right) = \mathcal{V}'(n_t)/\bar{A} \tag{2}$$

$$= \beta\mathcal{U}'(m_{t+1})\frac{1}{1 + \pi_{t+1}} \tag{3}$$

and

$$m_{t+1}(1 + \pi_{t+1}) - m_t = \delta\bar{k} \tag{4}$$

where $\bar{A} \equiv A\left(\bar{k}\right)$, $1 + \pi_{t+1} = p_{t+1}/p_t$, and $m_t = M_t/p_t$.

Answer

The first line is the static condition with $w = \bar{A} \equiv A\left(\bar{k}\right)$ and $s = m'(1 + \pi')$ substituted in. The second equation is the Euler equation with $c'^o = R's = m'$ plugged in. Finally, (4) is the government budget constraint.

---

4. [10 points] Assume now that the young do not consume; they save their entire income. Derive a difference equation for n_t:

$$\mathcal{V}'(n_t)/\bar{A} = \beta\mathcal{U}'\left(\bar{A}n_{t+1} - \delta\bar{k}\right)\frac{\bar{A}n_{t+1} - \delta\bar{k}}{\bar{A}n_t} \tag{5}$$

Answer

Now we have $s = m'(1 + \pi') = A\left(\bar{k}\right)n$ and therefore, using the resource constraint and old budget constraint

$$c'^o = m' = A\left(\bar{k}\right)n' - \delta\bar{k} \tag{6}$$

we have

$$1 + \pi' = \frac{A\left(\bar{k}\right)n}{m'} = \frac{A\left(\bar{k}\right)n}{A\left(\bar{k}\right)n' - \delta\bar{k}} \tag{7}$$

Substitute this into the Euler equation to obtain (5).

---

5. [10 points] Characterize the planner's solution, again assuming the only the old consume. The planner takes $\bar{k}$ as given.

   Does the steady state ever coincide with the equilibrium steady state (for any planner's "discount factor" ω)? Explain.

Answer

The planner solves

$$\max \sum_{t=0}^{\infty} -\omega^t \mathcal{V}(n_t) + \omega^{t-1} \beta \mathcal{U}\left(\bar{A} n_t - \delta \bar{k}\right) \tag{8}$$

with FOC

$$\omega \mathcal{V}'(n_t) = \beta \mathcal{U}'\left(\bar{A} n_t - \delta \bar{k}\right) \bar{A} \tag{9}$$

Unsurprisingly, the planner chooses a constant n that satisfies a standard static FOC.

Note that the equilibrium steady state is *never* Pareto efficient. The planner places "rate of return" $1/\omega > 1$ on the consumption of the old. In equilibrium, the transfer from young to old utility works through saving, which has a negative rate of return (because seignorage must be positive).

---

2 Two Period Capital Depreciation

Demographics: There is a representative agent of mass 1 who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$.

Endowments: k_0 at $t = 0$.

Technology: $f(k_t) = c_t + i_t$ with $k_{t+1} = i_t + i_{t-1}$. In words: capital fully depreciates after two periods of use.

Questions:

1. [10 points] Write down the planner's Bellman equation.

   Answer ______________________________

   Define two auxiliary state variables:[2] $z_t = i_{t-1}$ and $x_t = i_{t-2}$. Then $k_t = z_t + x_t$. The laws of motion are: $x_{t+1} = z_t$ and $z_{t+1} = f(k_t) - c_t$.

   Bellman: $V(x, z) = \max_c u(c) + \beta V(z, f(z + x) - c)$.

   ---

2. [10 points] Derive the planner's first-order conditions and define a solution to the planner's problem in sequence language.

   Answer ______________________________

   FOC:

[2] Based on an example due to Per Krusell.

- $u'(c) = \beta V_2(.')$. $V_1 = \beta V_2(.') f'(x+z)$.
- $V_2 = \beta V_1(.') + \beta V_2(.') f'(x+z)$.

Euler: $u'(c) = \beta u'(c') f'(k') + \beta^2 u'(c'') f'(k'')$.

Solution: Sequences $\{c_t, k_{t+1}\}$ that solve the Euler equation and the resource constraint. With boundary conditions k_0 given and $\lim \beta^t u'(c_t) k_t = 0$.

End of exam.