Midterm Exam. Econ720. Fall 2020

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- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 1:15 hours.
- The total number of points is 100.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

Special instructions for take-home exams:

- This exam is open book. You may consult books and notes (but not persons).
- To ask questions during the exam: call 919-886-6885, or text or facetime Lutz Hendricks at hendricks.lutz@gmail.com.

Your name: ____________________________________

1 Three Period OLG Model

Demographics: Time is discrete and goes on forever. In each period, a unit mass of young households are born. Each lives for 3 periods (young, middle aged, old).

Preferences: $u\left(c_t^y\right)+\beta u\left(c_{t+1}^m\right)+\beta^2 u\left(c_{t+2}^o\right)$ with $0<\beta<1$ and u strictly concave and increasing.

Endowments: Each young is endowed with h^y unit of human capital.

Technology:

- Young agents can use goods to augment their human capital according to $h_{t+1}^m = h^y + f\left(s_t\right)$ where f is strictly concave and increasing. Workers of middle and old age supply this human capital to firms who pay wages.
- Output is produced from labor only according to

$$Y_t = h_t^m L_t^m + h_{t-1}^m L_t^o = C_t + s_t \tag{1}$$

  where L_t^m (L_t^o) is the number of middle ages (old) persons employed and C_t is aggregate consumption. In words: each "hour" that a worker spends working supplies h_t^m units of labor to the firm, who then produces h_t^m units of output from it.

Markets: There are competitive markets for goods (numeraire), human capital rental (wage w), bonds b (gross interest rate R). Bonds are in zero net supply. Negative values of b means that a household is borrowing.

The budget constraints for generation t are

$$c_t^y = -b_{t+1}^m - s_t \tag{2}$$
$$c_{t+1}^m = w_{t+1} h_{t+1}^m + R_{t+1} b_{t+1}^m - b_{t+2}^o \tag{3}$$
$$c_{t+2}^o = w_{t+2} h_{t+1}^m + R_{t+2} b_{t+2}^o \tag{4}$$

Questions:

1. [13 points] Write down the household problem as a set of Bellman equations. The old "Bellman equation" is trivial (because the old make no choices), but write it down anyway.

   A: Bellman equations:

$$V^y\left(h^y\right) = \max_{b^m, s} u\left(-b^m - s\right) + \beta V^m\left(h^y + f\left(s\right), b^m\right) \tag{5}$$
$$V^m\left(h^m, b^m\right) = \max_{b^o} u\left(w h^m + R b^m - b^0\right) + \beta V^o\left(h^m, b^o\right) \tag{6}$$
$$V^o\left(h^m, b^o\right) = u\left(R b^o + w h^m\right) \tag{7}$$

2. [18 points] Write down the first order and envelope conditions. Derive and interpret the Euler equations.

   A: FOCs:

(a) Old: $V_b^o(.) = u'(c^o) R'$ and $V_h^o(.) = u'(c^o) w$.

(b) Middle aged: $u'(c^m) = \beta V_b^o(.) = \beta u'(c^o) R'$. Envelope: $V_b^m(.) = u'(c^m) R$ and $V_h^m(.) = u'(c^m) w + \beta V_h^o(.)$

(c) Young: $u'(c^y) = \beta V_b^m(.) = \beta R' u'(c^m)$ and

$$u'(c^y) = \beta V_h^m(.) f'(s) = f'(s) \left[\beta u'(c^m) w' + \beta^2 u'(c^o) w''\right] \tag{8}$$

(d) Interpretation: We have two standard Euler equations for bonds. Nothing new here. The new equation is (8). It says: Giving up a unit of young consumption buys $f'(s)$ units of human capital at middle and old ages. They earn wages that can be eaten.

3. [10 points] Derive and interpret

$$R_{t+1} = f'(s_t) \left[w_{t+1} + w_{t+2}/R_{t+2}\right] \tag{9}$$

A: Equating the two Euler equations for $u'(c^y)$ and replacing $u'(c^o)$ using the middle aged Euler equation gives (9). It says in words that the two assets have the same rate of return.

Alternatively, observe that agents choose s to maximize lifetime income:

$$s = \text{argmax} \left[h^y + f(s_t)\right] \left[w_{t+1} + w_{t+2}/R_{t+2}\right] / R_{t+1} - s_t \tag{10}$$

4. [14 points] Define a competitive equilibrium in sequence language.

A: CE:

Sequences $\{c_t^y, c_t^m, c_t^o, s_t, b_t^m, b_t^o, h_t^m, L_t^m, L_t^o, C_t\}$ and $\{w_t, R_t\}$ that satisfy:

(a) Consumption, s, human capital, and bond holdings solve the household problem, given prices.

(b) Firm first order conditions: $w_t = 1$

(c) Market clearing: goods (R.C.) where $C_t = c_t^y + c_t^m + c_t^o$; labor ($L_t^m = L_t^o = 1$); bonds ($b_t^m + b_t^o = 0$).

We have 12 objects and 13 equations (7 for the household, 1 firm, 4 market clearing, definition of C).

5. [15 points] Now consider the same economy decentralized using Arrow-Debreu trading. State the household problem.

A: Arrow-Debreu household solves $\max u(c_t^y) + \beta u\left(c_{t+1}^m\right) + \beta^2 u\left(c_{t+2}^o\right)$ subject to the budget constraint

$$p_t (c_t^y + s_t) + p_{t+1} c_{t+1}^m + p_{t+2} c_{t+2}^o = \left[h^y + f(s_t)\right] \left[w_{t+1} + w_{t+2}\right] \tag{11}$$

6. [17 points] Derive the household's first-order conditions and show that

$$u'\left(c_t^y\right) = f'\left(s_t\right)\left[\beta u'\left(c_{t+1}^m\right) + \beta^2 u'\left(c_{t+2}^o\right)\right] \tag{12}$$

(which we also obtained from the sequential trading equilibrium). Hint: the firm's first-order condition is $w_t = p_t$ where p_t is the price of the good at t.

A: First order conditions:

$$u'\left(c_t^y\right) = \lambda_t p_t \tag{13}$$
$$\beta u'\left(c_{t+1}^m\right) = \lambda_t p_{t+1} \tag{14}$$
$$\beta^2 u'\left(c_{t+2}^o\right) = \lambda_t p_{t+2} \tag{15}$$
$$p_t = f'\left(s\right)\left[w_{t+1} + w_{t+2}\right] \tag{16}$$

Eliminating λ_t we get two standard Euler equations with "interest rates" p_t/p_{t+1}. Applying $w_t = p_t$ (the firm's FOC), we get the more intuitive Euler equation (12).

7. [13 points] Define a competitive equilibrium.

A: Competitive equilibrium:

Sequences $\{c_t^y, c_t^m, c_t^o, s_t, h_t^m, L_t^m, L_t^o\}$ and $\{p_t, w_t\}$ that satisfy

(a) household (5 equations)
   i. firm: $w_t = p_t$
   ii. market clearing: goods (RC), labor ($L_t^m = L_t^o = 1$).

End of exam.