Final Exam. Econ720. Fall 2023

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- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 120 minutes.
- The total number of points is 120.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Short Questions

1. [10 points] Consider a firm with market value

$$V(t) = \mathbb{E}\int_0^\infty e^{-r\tau}\pi(t+\tau)\,d\tau \tag{1}$$

where $\pi(t+\tau) = \bar{\pi}$ until a discrete random event occurs. There are two types of random events. With Poisson arrival rate ρ, profits permanently fall to zero. With Poisson arrival rate η, profits permanently change by factor $\lambda > 0$.

Solve for the value of the firm. Interpret the solution for the case where $\eta = 0$.

Hint: Write out a differential equation that governs the value of the firm of the form $rV = \pi + \dot{V} + \dots$. Assume and verify that V is proportional to π.

Answer

The standard pricing equation implies

$$rV = \pi + \dot{V} - \rho V + \eta(\lambda V - V) \tag{2}$$

With $\dot{V} = 0$, this implies

$$V = \frac{\pi}{r + \rho - (\eta\lambda - 1)} \tag{3}$$

With $\eta = 0$, losing profits acts like higher discounting.

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2. [5 points] In a Recursive Competitive Equilibrium we typically have a consistency condition for the law of motion for the aggregate state. What does this condition say in words? And why is it needed?

Answer

The consistency condition says: If households expect the aggregate state to evolve according its law of motion, then the optimal household decisions aggregate to produce that law of motion. In other words, rational expectations are correct in equilibrium.

We need the condition to ensure that household expectations are indeed rational (correct, in the case of perfect foresight).

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3. [5 points] Many monetary models imply the Friedman Rule for optimal monetary policy (set the nominal interest rate to zero). What is the general intuition for this result?

Answer

Money is a good that is produced at zero marginal cost, but has positive value (e.g., in preferences or in transactions). The optimal price for a good is its marginal cost – zero in this case. Money is free to hold when the nominal interest rate is zero.

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2 Wealth Externality

We study a standard growth model in continuous time where households get utility from holding wealth.

Demographics: There is a single representative household who lives forever.

Preferences:

- Households value consumption (c) and wealth (W) according to

$$\int_0^\infty e^{-\rho t} u(c_t, W_t/\bar{W}_t) dt \tag{4}$$

  with

$$u(c, W/\bar{W}) = \frac{c^\sigma}{\sigma} \times \frac{\left(W/\bar{W}\right)^\lambda}{\lambda} \tag{5}$$

  where $0 < \sigma, \lambda < 1$.

- Wealth consists of capital and bonds: $W_t = K_t + B_t$.
- $\bar{W}_t$ denotes average wealth, which the household takes as given.

Technologies: A single good is produced from capital according to $Y_t = AK_t$. The resource constraint is $\dot{K}_t = Y_t - c_t - G_t$. G denotes useless government spending.

Endowments: At date $t = 0$ the household is endowed with K_0 and bonds B_0. Bonds are issued by the government and pay interest $r = A$.

Government: The government taxes all income at rate τ. The flow budget constraint is $\dot{B}_t = G_t - \tau r B_t - \tau Y_t$. Government spending is a constant fraction of output: $G_t = \phi Y_t$.

Markets: There are competitive markets for goods (numeraire), capital rental (price r), and bonds (interest rate r).

Questions

1. [16 points] For the household:
   - Write down the household's current value Hamiltonian. State the states and controls. The household receives capital and bond income in the amount of $(1 - \tau) r W$. There is no labor income.
   - Derive the household's first-order conditions. It is best not to eliminate the costate.
   - Define a solution to the household problem (objects and equations).

Answer

The household budget constraint is $\dot{W} = (1-\tau)AW_t - c_t$. Note that wealth is the only state variable needed (not K and B separately).

Hamiltonian:

$$H = u(c, W/\bar{W}) + \mu\left((1-\tau)rW - c\right) \tag{6}$$

First-order conditions:

$$u_c = \mu \tag{7}$$

$$\dot{\mu} = \rho\mu - \mu(1-\tau)r - u_W \tag{8}$$

Solution: Functions of time c_t, W_t, μ_t that satisfy the 2 FOCs, the budget constraint, and a TVC.[1]

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2. [11 points] Define a competitive equilibrium (objects and equations).

Answer

Equilibrium: 8 objects, $\mu_t, c_t, K_t, B_t, W_t, \bar{W}_t, G_t, r_t$, that satisfy 9 equations:

- 3 household conditions
- government: budget constraint and $G = \phi Y$.
- Firm: $r = A$.
- market clearing: goods (resource constraint), capital rental (implicit in notation), and bonds (implicit).
- Identity $W = B + K$
- $\bar{W} = W$.

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3. [11 points] Derive the (equilibrium) Euler equation

$$g(c) = \frac{(1-\tau)A - \rho + \frac{\lambda}{\sigma}\frac{c}{W}}{1-\sigma} \tag{9}$$

where we impose the equilibrium condition $W/\bar{W} = 1$, so that $g(c) = (\sigma - 1)\, g(u')$. Explain why wealth in the utility function ($\lambda > 0$) increases consumption growth relative to the standard growth model.

Answer

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[1] This question is ased on Liutang and Mengying (2008).

Divide the $\dot{\mu}$ equation by μ

$$g(\mu) = \rho - (1-\tau)\, r - u_W/u_c \tag{10}$$

Note that $u_W = \lambda u/W > 0$ and $u_c = \sigma u/c > 0$, so that

$$\frac{u_W}{u_c} = \frac{\lambda}{\sigma}\frac{c}{W} \tag{11}$$

Next, impose $g(\mu) = g(u_c)$ to arrive at

$$g(u_c) = \rho - (1-\tau)r - \frac{\lambda}{\sigma}\frac{c}{W} \tag{12}$$

In equilibrium, $W = \bar{W}$, so that $g(u_c) = (\sigma - 1)\, g(c)$.

Interpretation: When $\lambda = 0$, this is a standard Euler equation. The growth rate of consumption equals the (after tax) interest rate minus the discount rate divided by the curvature parameter. Households distribute lifetime income over their entire lifetimes. The slope of the age consumption profile depends on the tug between interest (postpone) and discounting (consume early).

The new part is the c/W term. Having wealth in utility gives an additional incentive to save. Accumulating wealth raises future marginal utility.

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4. [8 points] Assume that the government does not issue bonds and sets G_t to balance the budget in each period. Show that the balanced growth rate of output and consumption ($\bar{g}$) is given by

$$\bar{g}\left\{1 + \frac{\lambda}{\sigma} - \sigma\right\} = (1-\tau)A\left\{1 + \frac{\lambda}{\sigma}\right\} - \rho \tag{13}$$

Answer

Let the balanced growth rate be $\bar{g}$. Household budget constraint: $g(K) = A(1-\tau) - \frac{c}{K}$. It follows that $\bar{g} = g(K) = g(c)$ and $c/K = A(1-\tau) - \bar{g}$ (which therefore must be positive for a balanced growth path to exist).

Balanced growth: $W/\bar{W} = 1$, so that $g(u_c) = (\sigma - 1)\, g(c)$. Substitute into the Euler equation (9) to obtain

$$\bar{g}(\sigma - 1) = \rho - (1-\tau)\, A - \frac{\lambda}{\sigma}\left((1-\tau)\, A - \bar{g}\right) \tag{14}$$

Collect the $\bar{g}$ terms on the LHS to obtain (13).

Note: When $\lambda = 0$ this boils down to

$$\bar{g} = \frac{(1-\tau)\, r - \rho}{1 - \sigma} \tag{15}$$

which is the Euler equation from the standard growth model.

From (9) we can derive again that higher λ increases the balanced growth rate. Brute force computing $\partial g/\partial\lambda$ yields $\frac{(1-\tau)A-\bar{g}}{\sigma Q}$ where $Q = 1 = \lambda/\sigma - xc > 0$. We know that $(1-\tau)\,A > \bar{g}$ so that $c/K > 0$. The intuition is (again) that there is an additional incentive to save.

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3 Guvenen, Kuruscu and Ozkan

Consider a household who lives for T periods. At birth, he is endowed with human capital h_1 and assets a_1.

The household maximizes the discounted utility from consumption c and leisure $1-n$, where n denotes work time

$$\sum_{t=1}^{T} \beta^t u\left(c_t, 1-n_t\right) \tag{16}$$

The household faces the budget constraint

$$c_t + a_{t+1} = \left(1-\tau\left(y_t\right)\right) y_t + R a_t \tag{17}$$

where a denotes assets, y is labor earnings before tax

$$y_t = w h_t n_t - C\left(Q_t\right) \tag{18}$$

$\tau\left(y\right)$ is a labor income tax, R is the gross interest rate, w is the wage rate, h is human capital.

Human capital is accumulated according to

$$h_{t+1} = h_t + Q_t \tag{19}$$

where Q_t is human capital investment and $C\left(Q_t\right)$ is its convex cost.

The household chooses sequences $\{c_t, n_t, a_t, h_t, Q_t\}$.

Questions

1. [10 points] State the household's dynamic program. It is helpful to substitute out c using the budget constraint, but keep (18) and (19) as separate constraints with their own multipliers. The controls are now h', a', Q, y, n.

   Answer

   The states are human capital, assets, and age (this is a finite horizon problem). The Bellman equation is given by

$$V\left(h,a,t\right) = \max_{h',a',Q,y,n} u\left(\left[1-\tau(y)\right] y + Ra - a', 1-n\right) + \beta V\left(h', a', t+1\right) \tag{20}$$

$$+\lambda\left(whn - C\left(Q\right) - y\right) \tag{21}$$

$$+\gamma\left(h + Q - h'\right) \tag{22}$$

   It is easier to keep the budget constraint as a separate constraint with another multiplier. One can, of course, substitute out all the constraints, but then the FOCs become messy.

2. [20 points] Derive the first-order conditions and envelope equations. It is useful to define $\hat{\tau}(y) = 1 - \tau(y) - \tau'(y)y$ which is the derivative of after tax income with respect to y. Explain in words what the optimality conditions mean.

Answer

FOC:

$$\beta V_h(.') = \gamma \tag{23}$$

$$u_c = \beta V_a(.') \tag{24}$$

$$\lambda C'(Q) = \gamma \tag{25}$$

$$u_c \hat{\tau}(y) = \lambda \tag{26}$$

$$u_l = \lambda w h \tag{27}$$

In words:

- A unit of h today becomes a unit of h' tomorrow
- A unit of c can be made into a unit of a'.
- A unit of h allows the household to reduce human capital investment by $C'(Q)$, which is valued at λ (the value of goods today).
- A unit of goods (income) yields the marginal after tax income $\hat{\tau}(y)$ and may be eaten.
- A standard static condition: giving up a unit of leisure earns wh, which may be eaten (after tax).

Envelope

$$V_a = u_c R \tag{28}$$

$$V_h = \lambda w n + \gamma \tag{29}$$

In words:

- A unit of assets pays R which may be eaten.
- A unit of h earns wn, which may be eaten. It also gives a unit of h' tomorrow (at value γ).

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3. [12 points] Derive the consumption Euler equation and the static labor-leisure condition. Explain the latter in words.

Answer

We get a standard Euler equation $u_c = \beta R' u_c(.')$, which is not distorted by taxes. and a standard static condition for labor-leisure choice

$$u_l / u_c = w h \hat{\tau}(y) \tag{30}$$

which says in words: The MRS between consumption and leisure is the wage per unit of time (wh) but after tax.

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4. [12 points] The closed form solution for human capital investment is given by (you need not derive this)

$$C'(Q_t) = \sum_{j=1}^{T-t} R^{-j} \frac{\hat{\tau}(y_{t+j})}{\hat{\tau}(y_t)} w n_{t+j} \tag{31}$$

Explain what (31) says in words. Based on (31), how would you expect the following to affect human capital investment:

(a) a flat income tax $\tau(y) = \bar{\tau}$;

(b) a progressive income tax where $\tau'(y) > 0$. Hint: In this case, $\hat{\tau}(y)$ is decreasing in y as richer households get to keep a smaller share of pre-tax earnings.

Answer

In words: the marginal cost of another unit of human capital equals the present discounted value of the (marginal) after tax incomes it generates over an agent's remaining lifetime.

Tax effects:

(a) If leisure were exogenous, the flat tax would not affect human capital investment as the $\hat{\tau}$ terms cancel. However, the flat income tax would reduce labor supply (by the static condition) and therefore human capital investment.

(b) A progressive that has an additional effect. For young agents, who do most of the investment, future earnings are higher than current earnings, so that the tax wedge $\hat{\tau}(y_{t+j})/\hat{\tau}(y_t) < 1$, which further reduces investment.

Note on how to derive (31): Start from

$$\gamma = \beta(\gamma' + \lambda' w' n') \tag{32}$$

Then

$$\frac{\gamma}{\lambda} = \beta\left(\frac{\gamma'}{\lambda'}\frac{\lambda'}{\lambda} + \frac{\lambda'}{\lambda} w' n'\right) \tag{33}$$

Next, from the Euler equation:

$$\beta\frac{\lambda'}{\lambda} = \frac{\beta u_c(.')}{u_c}\frac{\hat{\tau}'}{\hat{\tau}} = \frac{1}{R'}\frac{\hat{\tau}'}{\hat{\tau}} \tag{34}$$

Therefore,

$$C'(Q_t) = \frac{\gamma_t}{\lambda_t} = \frac{\hat{\tau}(y_{t+1})}{\hat{\tau}(y_t)}\frac{1}{R_{t+1}}\left[w n_{t+1} + C'(Q_{t+1})\right] \tag{35}$$

Assume that wages are constant and iterate forward to obtain (31).

End of exam.