Final Exam. Econ720. Fall 2022

Professor Lutz Hendricks. UNC.

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 180 minutes.
- The total number of points is 120.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

---

1 Growth with Two Capital Goods

Demographics: A single infinitely lived household.

Preferences: $\int_0^\infty e^{-\rho t} u(c_t) dt$ with $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

Endowments: k_0, h_0 at $t = 0$.

Technologies:

- Sector 1 produces consumption and capital of type k: $G(k_1, h_1) = c + I_1$ where $I_1 = \dot{k} + \delta k$.
- Sector 2 produces capital of type h: $H(k_2, h_2) = I_2$ where $I_2 = \dot{h} + \delta h$.
- G and H are constant returns to scale.
- $h = h_1 + h_2$ and $k = k_1 + k_2$.

Government: The government taxes capital income at rates τ_k and τ_h. It rebates revenues as a lump-sum transfer T. The budget constraint is given by $T = \tau_k q_k k + \tau_h q_h h$.

Market arrangements: There are rental markets for the two types of capital with rental prices q_k and q_h, respectively. Households own the capital. There are also competitive markets for the two goods. The consumption good is the numeraire. The price of h is p.

Questions:

1. [12 points] State the household's budget constraint and Hamiltonian. Be careful about what units prices are in. Derive the household's first-order conditions.

Answer

The household's state variables are k and h. The budget constraint is given by

$$c + I_1 + pI_2 = (1 - \tau_k) q_k k + (1 - \tau_h) q_h h + T \tag{1}$$

Hamiltonian:

$$H = u(c) + \lambda[I_1 - \delta k] + \mu[I_2 - \delta h] \tag{2}$$

where c is given by the b.c. FOC:

$$I_1 : u'(c) = \lambda \tag{3}$$

$$I_2 : u'(c)p = \mu \tag{4}$$

$$k : \dot{\lambda} = \rho\lambda + u'(c)(1 - \tau_k) q_k - \lambda\delta \tag{5}$$

$$h : \dot{\mu} = \rho\mu + u'(c)(1 - \tau_h) q_h - \mu\delta \tag{6}$$

This can also be set up with total assets $a = k + ph$ as the state and k or h as a control. But it's arguably more complicated.

2. [12 points] Derive the household's Euler equation $g(c) = (r - \rho)/\sigma$ with

$$r = (1 - \tau_k)q_k - \delta = \frac{(1 - \tau_h)q_h}{p} + \frac{\dot{p}}{p} - \delta \tag{7}$$

Answer

The first Euler equation follows directly from first-order conditions using the standard argument. The second Euler equation follows from

$$-g(\mu) = \sigma g(c) - \dot{p}/p = \frac{(1 - \tau_h)q_h}{p} - \delta - \rho \tag{8}$$

---

3. [12 points] Define a competitive equilibrium.

Answer

Functions of time $c, k, h, k_1, k_2, h_1, h_2, T$ and p, q_k, q_h that satisfy

(a) Household: 2 Euler equations and budget constraint with boundary conditions.

(b) Firm: $q_k = G_1 = pH_1$. $q_h = G_2 = pH_2$.

(c) Government budget constraint.

(d) Market clearing

  i. goods: 2 RC

  ii. rental markets: implicit in notation

(e) Identities: $k = k_1 + k_2$ and $h = h_1 + h_2$.

---

4. [12 points] Derive 4 equations that solve for the balanced growth values of g, z_1, z_2, r where g is the balanced growth rate of (c, k, h) and $z_i = k_i/h_i$. Note that p is constant on the balanced growth path. Remember that, with constant returns to scale, marginal products are functions of z_i.

Answer

Balanced growth path: g, z_1, z_2, r that satisfy:

$$g = \frac{r - \rho}{\sigma} \tag{9}$$

$$r = (1 - \tau_k)G_1(1, z_1) - \delta \tag{10}$$

$$r = (1 - \tau_h)H_2(1, z_2) - \delta \tag{11}$$

$$\frac{G_1}{G_2} = \frac{H_1}{H_2} \tag{12}$$

---

5. [10 points] For the special case where $H(k_{2,}, h_2) = Bh_2$ show that taxes on sector 1 do not affect the balanced growth rate. What is the intuition for this result?

Answer

Now $H_2 = B$ so that $r = (1 - \tau_h)B - \delta$. The sector with the linear technology fixes the after-tax interest rate. Taxing sector 1 merely changes levels. z_1 adjusts to maintain equal after-tax interest rates in both sectors.

---

2 Asset Pricing Inclusive of Dividends

Consider the standard Lucas fruit tree model where the shares are priced inclusive of dividends.

Demographics: A single, representative household who lives forever in discrete time.

Preferences: $\mathbb{E} \sum_{t=0}^{\infty} \beta^t \mathcal{U}(c_t)$ where c is consumption and $\mathcal{U}$ is well behaved.

Endowments: At the beginning of time, the household owns zero bonds b and one tree k. In each period, the tree produces fruit d which evolves according to a Markov chain.

Technology: Fruit can only be eaten: $c_t = d_t$.

Markets:

- There are competitive markets for fruit (numeraire), trees (price p), and bonds (price 1; interest rate R).
- The only new item: trees are priced inclusive of dividends. That is, the household first buys or sells shares $(k' - k)$ and then receives dividends $d \times k'$.

Questions:

1. [9 points] Write out the household's Bellman equation (with budget constraint).

Answer

Bellman equation:[1]

$$V(k, b, d) = \max_{k', b'} \mathcal{U}(pk + Rb - b' - [p - d]\, k') + \beta \mathbb{E} V(k', b', d') \tag{13}$$

---

[1] NB: In the original question, I forgot the discount factor β. I graded answers with and without β as correct.

2. [20 points] Derive the Lucas asset pricing equations for bonds and trees.

Answer

FOCs

$$\mathcal{U}'(c) = \beta \mathbb{E} V_b(k', b', d') \tag{14}$$
$$\mathcal{U}'(c)(p-d) = \beta \mathbb{E} V_k(k', b', d') \tag{15}$$

Envelope:

$$V_b = \mathcal{U}'(c) R \tag{16}$$
$$V_k = \mathcal{U}'(c) p \tag{17}$$

Lucas equations:

$$\mathcal{U}'(c) = \beta \mathbb{E} \mathcal{U}'(c') R' \tag{18}$$
$$\mathcal{U}'(c) = \beta \mathbb{E} \mathcal{U}'(c') \frac{p'}{p-d} \tag{19}$$

---

3. [9 points] Show that the equilibrium share price is given by $p = \hat{p} - d$ where $\hat{p}$ is the share price in the model with ex dividend pricing that we studied in class. You do not need to derive the solution for that model.

Answer

An equilibrium is an allocation $\{c_t, k_t, b_t\}$ and prices $\{p_t, R_t\}$ that solve

- household: 2 x Euler + BC + boundary conditions
- market clearing: $b = 0$, $k = 1$, $c = d$.

All equilibrium conditions are the same as in the ex dividend model, except for the share price equation. But since

$$\frac{p'}{p-d} = \frac{\hat{p}' + d'}{\hat{p}} \tag{20}$$

that equation is also the same.

---

3 The Cake Eating Problem

This is a decision problem for a single agent who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$.

Endowments: x_0 units of consumption at $t = 0$.

Technology: The household eats the cake over time. $x_{t+1} = x_t - c_t$. $x_t \geq 0$ and $c_t \geq 0$ for all t.

Questions:

1. [5 points] Set up the household's Bellman equation.

   Answer

   $V(x) = \max_c u(c) + \beta V(x - c)$.

2. [5 points] Derive the Euler equation and explain it.

   Answer

   Euler: $u'(c) = \beta u'(c')$. This is the standard Euler equation for the case where the interest rate is 0.

3. [12 points] Assume that $u(c) = \ln(c)$. Derive a closed form solution for c_t. Recall that $\sum_{\tau=0}^{t-1} \beta^t = \frac{\beta^t - 1}{\beta - 1}$. Verify that your solution satisfies the transversality condition.

   Answer

   Now $c_{t+1}/c_t = \beta$ and therefore $c_t = c_0 \beta^t$. Then $x_t = x_0 - \sum_{\tau=0}^{t-1} c_\tau = x_0 - \sum_{\tau=0}^{t-1} \beta^t c_0 = x_0 - c_0 \frac{\beta^t - 1}{\beta - 1}$. With $\lim_{t \to \infty} x_t = 0$ we have $c_0 = (1 - \beta) x_0$, which makes intuitive sense.

   The TVC is $\lim_{t \to \infty} \beta^t u'(c_t) x_t = 0$. In the log case, $\beta^t u'(c_t) = c_0$ and therefore the TVC implies that $\lim_{t \to \infty} x_t = 0$.

End of exam.