Final Exam. Econ720. Fall 2021

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- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 180 minutes.
- The total number of points is 180.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Equity Financed Firms

Demographics: There is a representative household of unit mass who lives forever.

Preferences: $\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}u(c_t)$.

Endowments: k_0 at $t=0$. One unit of work time in each period.

Technologies:

$$k_{t+1}=\mathcal{F}(k_t,l_t)+(1-\delta)k_t-c_t-g_t \tag{1}$$

$\mathcal{F}$ has constant returns to scale.

Government: The government consumes the stochastic sequence g_t which follows a finite Markov chain. The government levies lump-sum taxes τ_t on households. The budget is balanced.

Markets:

- There are markets for goods (numeraire), labor rental (w), one period bonds (gross return R), and firm shares (p).
- Bonds are only traded by households. They are in zero net supply.
- There is a representative firm who lives forever. It owns the capital stock and pays out dividends given by $d=\mathcal{F}(k,l)+(1-\delta)k-wl-k'$. Dividends are allowed to be negative when the firm requires outside funds for investment.
- The firm maximizes its market value or, equivalently, its share price (p).

Questions:

1. [8 points] We consider a Recursive Competitive Equilibrium. What is the aggregate state?

   Answer

   The aggregate state $S_t=(K_t,g_t)$ is simply the vector of the individual states where the household does not have a non-trivial one (in equilbrium, households own a fixed number of shares). To distinguish aggregate from individual firm capital, I use K here.

2. [24 points] Write down the household problem and define a solution in recursive language. Derive the Lucas asset pricing equation

$$\mathbb{E}\left\{\frac{\beta u'(c')}{u'(c)}\frac{p'+d'}{p}\right\}=1 \tag{2}$$

   Note that this implies a share price of

$$p_t=\mathbb{E}\sum_{j=0}^{\infty}\frac{\beta^{j}u'(c_{t+j})}{u'(c_t)}d_{t+j} \tag{3}$$

Answer

This is exactly as in the Lucas fruit tree model. Solution is a value function $\mathcal{V}(q, b, S)$ and policy functions $\mathcal{C}(q, b, S)$ and $\mathcal{Q}(q, b, S)$ that solve the household problem in the usual sense.

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3. [15 points] Write down the firm's problem in recursive language. Remember that the firm maximizes its value, given by (3).

Answer

The firm maxmizes its value p. Its Bellman equation is

$$V(k, S) = \max_{k', l} \mathcal{F}(k, l) + (1 - \delta) k - wl - k' + \mathbb{E}\left\{\frac{\beta u'(c')}{u'(c)} V(k', S')\right\} \tag{4}$$

In equilibrium, $V(k, S) = p$.

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4. [20 points] Derive the firm's first order conditions and eliminate value function derivatives. Define a solution in recursive language. Show that the Lucas asset pricing equation

$$1 = \mathbb{E}\{MRS \times [\mathcal{F}_k(k', l') + 1 - \delta]\} \tag{5}$$

holds, where MRS is the marginal rate of substitution between today and tomorrow. If you could not solve #3, you can solve this question in sequence language (for reduced credit).

Answer

For labor: $\mathcal{F}_l = w$ as usual.

For capital: $\mathbb{E} MRS \times V_k(k', g') = 1$ with envelope condition $V_k = \mathcal{F}_k + 1 - \delta$. And thus (5) holds.

A solution consists of a value function V and policy functions for k' and l that solve the Bellman equation in the usual sense.

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5. [14 points] Show that the firm chooses capital according to

$$R' = \mathcal{F}_k(k', l') + 1 - \delta \tag{6}$$

where R is the risk-free bond rate. Explain why the firm's decision rule is the same as in an economy without uncertainty.

Answer

Note that $\mathcal{F}_k$ is deterministic and therefore

$$1 = \mathbb{E}\{MRS\} \times [\mathcal{F}_k(k', l') + 1 - \delta] \tag{7}$$

In equilibrium, this means that the firm uses the risk-free rate $(R')^{-1} = \mathbb{E}\{MRS\}$ to discount the future when choosing capital.

Intuition: There is consumption risk. In general, the firm would have to consider the correlation between the marginal product of capital and the MRS. But in this model the aggregate shock does not affect the marginal product of capital. Hence that correlation is zero.

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6. [14 points] Define a Recursive Competitive Equilibrium.

Answer

Objects:

- Household value function and policy functions
- Firm value function and policy functions
- Price functions

Equilibrium conditions:

- Household (as usual)
- Firm (as usual)
- Market clearing.

Market clearing:

- Goods: (1)
- Bonds: $\mathcal{B}(q, b, S) = 0$
- Shares: $\mathcal{Q}(q, b, S) = 1$
- Labor: $\mathcal{L}(k, S) = 1$

Consistency: Let the aggregate law of motion be $K' = \mathcal{G}(S)$ and the firm's policy function be $k' = \mathcal{K}(k, S)$. Then consistency requires $\mathcal{G}(S) = \mathcal{K}(K, S)$.

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2 Growth Model With Human Capital

Demographics: A single infinitely lived household.

Preferences: $\int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt$ with $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

Endowments: Physical capital $k\left(0\right)$ and human capital $h\left(0\right)$ at $t = 0$.

Technologies:

- Sector 1 produces consumption and capital goods:

$$G(k_1\left(t\right), h_1\left(t\right)) = c\left(t\right) + I_1\left(t\right) \tag{8}$$

  where $I_1\left(t\right) = \dot{k}\left(t\right) + \delta k\left(t\right)$ and $k\left(t\right) = k_1\left(t\right) + k_2\left(t\right)$.

- Sector 2 produces human capital:

$$H(k_2\left(t\right), h_2\left(t\right)) = I_2\left(t\right) \tag{9}$$

  where $I_2\left(t\right) = \dot{h}\left(t\right) + \delta h\left(t\right)$ and $h\left(t\right) = h_1\left(t\right) + h_2\left(t\right)$.

- G and H are constant returns to scale.

Government: The government taxes capital income and human capital (labor) income. It rebates revenues as a lump-sum transfer $T\left(t\right)$. The budget constraint is given by

$$T\left(t\right) = \tau_k q_k\left(t\right) k\left(t\right) + \tau_h q_h\left(t\right) h\left(t\right) \tag{10}$$

where q_k and q_h are the rental prices of physical and human capital, respectively.

Market arrangements:

- There is a representative firm in each sector that rents physical and human capital from households.
- Good 1 is the numeraire.
- The price of good 2 is p.

The household's budget constraint is given by

$$c\left(t\right) + I_1\left(t\right) + p\left(t\right) I_2\left(t\right) = (1 - \tau_k) q_k\left(t\right) k\left(t\right) + (1 - \tau_h) q_h\left(t\right) h\left(t\right) + T\left(t\right) \tag{11}$$

Questions:

1. [23 points] Derive the household's first-order conditions. Throughout, assume an interior solution where $I_1(t), I_2(t) > 0$.

   Answer

   Hamiltonian:
$$H = u(c) + \lambda[I_1 - \delta k] + \mu[I_2 - \delta h] \tag{12}$$
   where c is given by the budget constraint. Hence we have
$$\frac{\partial H}{\partial h} = u'(c)(1-\tau_h) q_h - \mu\delta \tag{13}$$

   FOC:
$$I_1 : u'(c) = \lambda \tag{14}$$
$$I_2 : u'(c)p = \mu \tag{15}$$
$$k : \dot{\lambda} = \rho\lambda - u'(c)(1-\tau_k)q_k + \lambda\delta \tag{16}$$
$$h : \dot{\mu} = \rho\mu - u'(c)(1-\tau_h)q_h + \mu\delta \tag{17}$$

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2. [23 points] Derive the household's Euler equation $g(c) = (r-\rho)/\sigma$ with
$$r = (1-\tau_k)q_k - \delta = \frac{(1-\tau_h)q_h}{p} + \frac{\dot{p}}{p} - \delta \tag{18}$$
   Expain (18) in words.

   Answer

   The first Euler equation follows directly from first-order conditions using the standard argument. The second Euler equation follows from
$$-g(\mu) = \sigma g(c) - \dot{p}/p = \frac{(1-\tau_h)q_h}{p} - \delta - \rho \tag{19}$$
   In words: The first part is standard. The second part says that giving up p units of consumption buys one unit of good 2 (human capital). That has the usual payoff (after tax rental price net of depreciation). In addition, human capital experiences a capital gain or loss as p changes over time.

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3. [11 points] Now consider the balanced growth path (assume it exists). Show that c, k and h must grow at the same rates, while p, q_k, and q_h must be constant over time. Remember that, with constant returns to scale, marginal products are functions of $z_i \equiv k_i/h_i$.

Answer

From the good 1 RC: $g(c) = g(I_1) = g(k)$. From the good 2 RC and the budget constraint: $g(I_2) = g(h) = g(c) - g(p)$.

The Euler equation requires that q_k be constant over time. With CRS the firm's first-order conditions are standard: $q_k = G_1 = pH_1$. $q_h = G_2 = pH_2$. So z_1 and therefore also z_2 must be constant. That implies constant p.

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4. [20 points] Derive 5 equations that solve for the balanced growth values of g, p, z_1, z_2, r where g is the common balanced growth rate of (c, k, h).

Answer

Balanced growth path: g, z_1, z_2, r that satisfy:

$$g = \frac{r - \rho}{\sigma} \tag{20}$$

$$r = (1 - \tau_k) G_1(z_1, 1) - \delta \tag{21}$$

$$r = (1 - \tau_h) H_2(1, z_2) - \delta \tag{22}$$

$$p = \frac{G_1}{H_1} = \frac{G_2}{H_2} \tag{23}$$

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5. [8 points] For the special case where $H(k_{2,}, h_2) = Bh_2$ show that capital income taxes (τ_k) do not affect the balanced growth rate. What is the intuition for this result?

Answer

Now $H_2 = B$ so that $r = (1 - \tau_h)B - \delta$. The sector with the linear technology fixes the after-tax interest rate. Taxing sector 1 merely changes levels. z_1 adjusts to maintain equal after-tax interest rates in both sectors.

End of exam.