Final Exam. Econ720. Fall 2020

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 3 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

Special instructions for remotely administered exams:

- This exam is open book. You may consult books and notes.
- To ask questions during the exam contact Lutz Hendricks at 919-886-6885 or hendricks.lutz@gmail.com (phone, text, or facetime work best; email can be delayed).
- When you are done, scan your exam using a scanner or a scanning app. Photos are hard to read. Then submit by email to lhendri@email.unc.edu.

---

1 Money in the Utility Function

Demographics: There is a single representative household who lives forever.

Preferences: $\int_0^\infty e^{-\rho t} u(c_t, m_t)\, dt$ with $\rho > 0$ and u strictly concave and increasing in both arguments.

Endowments: At the beginning of time, the household is endowed with k_0 units of capital and M_0 units of money.

Technologies: Goods are produced according to

$$y = \mathcal{F}(k, l) - \delta k = c + \dot{k} + \mathcal{H}\left(\dot{M}/p\right) \tag{1}$$

where $\mathcal{F}$ has constant returns to scale. l is labor input. $\mathcal{H}$ is the cost of adjusting the household's money holdings with $\mathcal{H}' > 0$ and $\mathcal{H}'' > 0$. This is includes the direct cost of purchasing money, $\dot{M}/p$.

Government: The government costlessly prints money and hands it to households as lump-sum transfers.

Markets: There are competitive markets for money (numeraire), goods (price p), capital rental (q), and labor rental (w).

Questions:

1. [6 points] Write down the household's Hamiltonian. Hint: The budget constraint is given by
$$\dot{k} = w - c + rk + x - \mathcal{H}\left(\dot{M}/p\right) \tag{2}$$
where x is the government's lump-sum money transfer.

2. [8 points] Derive the first-order conditions
$$g(\mu) = g(u_c) = \rho - r \tag{3}$$
which is the standard Euler equation and
$$g(\lambda) = \rho + \pi - u_m/\lambda \tag{4}$$
$$\lambda = u_c \mathcal{H}'(z + \pi m) \tag{5}$$
where λ is the costate for m and $z \equiv \dot{m}$.

3. [4 points] Define a solution to the household problem.

4. [8 points] Define a competitive equilibrium. Assume that the government fixes the growth rate of the money supply $g(M) = \gamma$.

5. [8 points] Consider the steady state. Derive three expressions that jointly solve for k, c, and m.

6. [3 points] Is money super-neutral? What if utility is separable (as in $u(c, m) = \mathcal{U}(c) + \mathcal{V}(m)$)? Explain.

1.1 Solution

1. The household has two state variables: k and m. So we need to invent a second law of motion: $\dot{m} = z$ where z becomes a control. Note that

$$\frac{\dot{M}}{p} = \dot{m} + \pi m \tag{6}$$

   The Hamiltonian is then

$$H = u(c, m) + \lambda z + \mu [w - c + rk + x - \mathcal{H}(z + \pi m)] \tag{7}$$

   where π is inflation.

2. First-order conditions:

$$\frac{\partial H}{\partial z} = \lambda - \mu \mathcal{H}'(z + \pi m) = 0 \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial c} = u_c - \mu = 0 \tag{9}$$

   and for the states:

$$\dot{\lambda} = \rho \lambda - u_m + \mu \mathcal{H}'(z + \pi m)\pi \tag{10}$$

$$\dot{\mu} = \rho \mu - \mu r \tag{11}$$

   Then simplify.

3. Solution: Functions of time $c_t, k_t, m_t, z_t, \lambda_t$ that satisfy
   (a) first-order conditions (3), (5), and (4)
   (b) budget constraint
   (c) $\dot{m} = z$
   (d) boundary conditions: m_0, k_0 given and TVC.

4. Equilibrium:
   (a) Firm problem: static profit maximization with standard first-order conditions $q = \mathcal{F}_k$ and $w = \mathcal{F}_l$. Solution: k, l.
   (b) Government: $px = \dot{M}$ or $x = \dot{M}/p = \gamma m$.
   (c) Objects: $c, k, m, z, \lambda; l; x; w, r, q, \pi$
   (d) Equations:
      i. household: 5
      ii. firm: 2
      iii. government: 1

iv. goods market (RC)
v. labor market: $l = 1$
vi. identities: $r = q - \delta$ and $\pi = \gamma - g(m)$.

5. Steady state:

   (a) Euler equation implies $r = \rho$. This fixes the capital stock via $\mathcal{F}_k = r + \delta$.

   (b) Constant m requires $\pi = \gamma$.

   (c) We then get two equation that can be solved for c and m:

$$\mathcal{F}(k, 1) - \delta k = c + \mathcal{H}(\gamma m) \tag{12}$$

$$\frac{u_m}{\rho + \gamma} = u_c \mathcal{H}'(\gamma m) \tag{13}$$

6. It follows that money is not super-neutral. Faster money growth increases the cost of adjusting M (which is specified, perhaps unfortunately, in nominal terms). Without more information about functional forms, we cannot sign the effect of higher money growth on outcomes.

2 Growth and Detrending

Demographics: There is a representative dynasty of infinitely lived individuals. The size of the population is $N_t = (1 + \eta)^t$ where $\eta \geq 0$.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \rho u(1 - h_t)\right] \tag{14}$$

where c is consumption and h is hours worked. We assume log utility, $u = \ln$, but keep the general notation for clarity.

Technology:

$$N_t c_t + K_{t+1} = (1 + \gamma)^t e^{z_t} K_t^{\alpha} (N_t h_t)^{\phi} L^{1-\alpha-\phi} \tag{15}$$

where $\alpha, \phi \in (0, 1)$, $\alpha + \phi \in (0, 1)$, $\gamma \geq 0$ determines productivity growth, K is aggregate capital, L is aggregate land (in fixed supply), and z is a productivity shock that follows $z_{t+1} = \rho z_t + \epsilon_{t+1}$ with $\rho \in (0, 1)$ and $\epsilon \overset{\text{i.i.d}}{\sim} N(0, \sigma^2)$.

Endowments: At the beginning of time, households are endowed with $L = 1$ units of land and K_0 units of capital.

Questions:

1. [8 points] Show that the balanced growth rates are given by

$$g(c) = g(k) = \frac{\gamma + \phi\eta}{1-\alpha} - \eta \equiv \bar{g} \tag{16}$$

Explain the intuition why $\bar{g}$ can be negative, even when there is positive productivity growth ($\gamma > 0$).

2. [4 points] How would your answer change if we set $\phi = 1 - \alpha$? Explain why $\bar{g}$ no longer depends on the population growth rate η.

3. [13 points] State the planner's problem as a dynamic program, derive the optimality conditions, and define a solution (for now without detrending). Hint: Define $A_t = (1+\gamma)^t$ and $k = K/N$ and

$$F(k, h, A, N, z) = Ae^z k^\alpha h^\phi (L/N)^{1-\alpha-\phi} \tag{17}$$

and then keep (k, A, N, z) as states.

4. [5 points] Derive the Euler equation

$$u'(c) = \beta \mathbb{E} u'(c') \frac{\frac{\partial F}{\partial k}(k', h', A', N', z')}{1+\eta} \tag{18}$$

and explain it in words.

5. [8 points] Define $Q_t = (1+\bar{g})^t$ and $\tilde{c}_t = c_t/Q_t$ and $\tilde{k}_t = k_t/Q_t$. Use this to derive the detrended resource constraint

$$\tilde{c} + \tilde{k}'(1+\eta)(1+\bar{g}) = e^z \tilde{k}^\alpha h^\phi L^{1-\alpha-\phi} B \tag{19}$$

where B is constant on the balanced growth path.

6. [6 points] Write the planner's problem as a stationary dynamic program.

7. [10 points] Derive the Euler equation.

2.1 Solution[1]

1. Balanced growth rates:

   (a) By the time constraint: $g(h) = 0$.

   (b) From the resource constraint: $g(c) + \eta = g(K) = \gamma + \alpha g(K) + \phi\eta$. This implies (16).

   (c) If $\gamma = 0$ and $\phi = 1 - \alpha$, we are back to the standard $g(k) = 0$.

[1] Based on UCLA spring 2017.

(d) If γ is small and $\phi < 1-\alpha$, we also have $g(k) = 0$. Productivity growth has to be strong enough to counteract diminishing returns to reproducible factors.

2. Case $\phi = 1-\alpha$: The balanced growth rate is now independent of population growth: $\bar{g} = \frac{\gamma}{1-\alpha}$. We now have constant returns to capital and labor. Faster population growth implies faster growth in capital (one-for-one), but not faster growth is capital per worker. This is the standard exogenous growth model.

3. Planner's problem: The Bellman equation can be written as

$$V(k, A, N, z) = \max_{h,K'} u\left(F(k, h, A, N, z) - k'(1+\eta)\right) + \rho u(1-h) \tag{20}$$

$$+ \beta \mathbb{E} V\left(k', A(1+\gamma), N(1+\eta), z'\right) \tag{21}$$

(a) FOCs:

$$u'(c)(1+\eta) = \beta \mathbb{E} V_k(.') \tag{22}$$

$$\rho u'(1-h) = u'(c) \frac{\partial F}{\partial h} \tag{23}$$

(b) Envelope:

$$V_k = u'(c) \frac{\partial F}{\partial k} \tag{24}$$

4. Euler: follows. Interpretation: This is really the standard Euler equation, except that there is full depreciation. The term involving population growth reflects the fact that 1 unit of consumption today becomes $1/(1+\eta)$ units of capital tomorrow (per person).

5. Detrended resource constraint: Define $Q_t = (1+\bar{g})^t$. Divide both sides by $Q_t N_t$ to obtain

$$\frac{c_t}{Q_t} + \frac{k_{t+1}}{Q_{t+1}}(1+\eta)(1+\bar{g}) = e^{z_t}\left(\frac{k_t}{Q_t}\right)^{\alpha} h_t^{\phi} L^{1-\alpha-\phi} B \tag{25}$$

$$\equiv f\left(\tilde{k}_t, h_t, z_t\right) \tag{26}$$

where

$$B = (1+\gamma)^t Q_t^{\alpha-1} N_t^{\alpha+\phi-1} \tag{27}$$

is constant on the balanced growth path.

6. Bellman equation:

$$V\left(\tilde{k}, z\right) = \max_{\tilde{k}',h} \ln \tilde{c} + \ln Q_t + \rho \ln(1-h_t) + \beta \mathbb{E} V\left(\tilde{k}', z'\right) \tag{28}$$

subject to (25). With log utility, the Q_t term does not affect marginal utility and can be ignored.

7. First-order conditions:

$$(1+\eta)(1+\bar{g})\,u'(\tilde{c}) = \beta \mathbb{E} V_k(.') \tag{29}$$

$$u'(\tilde{c})\,f_h = \rho u'(1-h) \tag{30}$$

$$V_k = u'(\tilde{c})\,f_k \tag{31}$$

Euler equation:

$$u'(\tilde{c}) = \frac{\beta \mathbb{E} u'(\tilde{c}')\,f_k(.')}{(1+\eta)(1+\bar{g})} \tag{32}$$

3 Short Questions

Explain your answers in words. No math required.

1. [4 points] Consider a Lucas tree economy with $N > 5$ trees.
   - The dividends for tree j are given by $d_{j,t} = z_t + \varepsilon_{j,t}$ for $j > 1$ and by $d_{1,t} = 0.5z_t + 2\varepsilon_{1,t}$.
   - $z_t > 0$ and all $\varepsilon_{j,t} > 0$ are drawn i.i.d. from the same distribution each period.
   - Total consumption is given by $c_t = \sum_{j=1}^{N} d_{j,t}$.
   - Which tree or trees earn the largest risk premium?
2. [5 points] In the same economy, suppose we have J types of households. Each has mass $1/J$. Let's say they differ in preferences.
   - What would be the correct aggregate state variable for defining a Recursive Competitive Equilibrium?
   - Is there a way of simplifying the state for this case?

3.1 Solution

1. We don't need any math here. Let's think about large N. Then the idiosyncratic shocks $\varepsilon_{j,t}$ are not correlated with consumption growth and therefore not priced. The least risky tree is tree 1 because the correlation of its dividend with consumption growth is the smallest.
2. Generically, the state contains the joint distribution of the households over their states. In this case, this is simply the z_j holdings for all households. One could debate whether or not z_t and all $\varepsilon_{j,t}$ should also be states. One could define equilibrium either way.

   In this case we know that all trees $j > 1$ are ex ante identical (but not perfect substitutes!), so each household will hold the same number of shares. All we need to keep track of is therefore $d_{1,t}$ and $d_{2,t}$ for each type.

End of exam.