Final Exam. Econ720. Fall 2019

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 2 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Growth with many capital goods

Demographics: A representative household of unit mass who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$.

Technology:

- Output is produced from N types of capital inputs according to

$$f\left(\overrightarrow{k}_t\right) = \prod_{j=1}^{N} k_{j,t}^{\theta_j} = c_t + \sum_{j=1}^{N} B_j i_{j,t} \tag{1}$$

  with $\sum_j \theta_j = 1$. $\vec{k}$ is the vector of all the k_j. Note: It may help to think of the special case with two capital goods where the technology is $k_1^{\theta_1} k_2^{1-\theta_1}$. The more general case is still Cobb-Douglas, just with more inputs.

- Capital is accumulated according to

$$k_{j,t+1} = (1 - \delta_j) k_{j,t} + i_{j,t} \tag{2}$$

Endowments: $k_{j,0}$ at $t = 0$.

Markets: Competitive rental markets (at prices $q_{j,t}$) and purchase markets (at prices $p_{j,t}$) for capital. Competitive final goods market (numeraire).

Questions:

1. [12 points] State the planner's dynamic programming problem and define its solution in sequence language. Assume that investment is interior: $i_{j,t} > 0$.
2. [3 points] Interpret the planner's Euler equation.
3. [14 points] Define a competitive equilibrium in sequence language. Assume that each capital good is produced by a separate representative firm (as is the final output good).
4. [3 points] Do you expect the model to exhibit sustained growth (a growth rate that is positive as capital stocks go to infinity)? Explain.
5. [7 points] Derive the equilibrium input ratio of two capital goods $k_{j,t}/k_{v,t}$ as a function of the interest rate r and parameters.
6. [4 points] Show that the model implies an AK production structure when all δ_j are the same.

1.1 Answer:

1. Bellman

$$V\left(\overrightarrow{k}\right)=\max u\left(f\left(\overrightarrow{k}\right)-\sum_{j}B_{j}i_{j}\right)+\beta V\left(\overrightarrow{k}'\right) \tag{3}$$

where $\overrightarrow{k}'$ is given by (2). First-order conditions:

$$u'(c)\,B_{j}=\beta V_{j}\left(\overrightarrow{k}'\right) \tag{4}$$

Envelope:

$$V_{j}\left(\overrightarrow{k}\right)=u'(c)\,f_{j}\left(\overrightarrow{k}\right)+\beta V_{j}\left(\overrightarrow{k}'\right)(1-\delta_{j}) \tag{5}$$

Implied Euler equation:

$$u'(c)\,B_{j}=\beta u'(c')\left[f_{j}\left(\overrightarrow{k}'\right)+(1-\delta_{j})\,B_{j}\right] \tag{6}$$

Solution: $\left\{c_{t},\overrightarrow{k}_{t},\overrightarrow{i}_{t}\right\}$ that solve Euler equation, resource constraints, initial conditions, TVC.

2. Interpretation of Euler: Give up B_j units of consumption to get one unit of type j capital next period. This produces f_j units of consumption; eat that. Then convert the remaining capital back to consumption at rate B_j and eat that. This is exactly what happens with one capital good when $B_1 = 1$.

3. Equilibrium:

   (a) Household: All capital goods produce the same rate of return; call that R. Then the household problem is standard with Euler equation $u'(c) = \beta u'(c')\,R'$ and budget constraint $a' = Ra - c$ where $a = \sum_j p_j k_j$.

   (b) Firm: Static profit maximization yields $p_j = B_j$ and $q_{j,t} = f_j\left(\overrightarrow{k}\right) = \theta_j f\left(\vec{k}\right)/k_j$.

   (c) Market clearing:

      i. capital rental: implicit

      ii. goods: RC

      iii. capital purchase: laws of motion for capital

      iv. identity: $a = \sum_j p_j k_j$.

      v. identity: $R_{t+1} = \frac{q_{j,t+1}+(1-\delta_j)p_{j,t+1}}{p_{j,t}}$

   (d) Objects: $\left\{a_t, c_t, \overrightarrow{k}, \overrightarrow{i}_t, R_t, q_{j,t}, p_{j,t}\right\}$.

4. The short answer is "yes" because the model exhibits constant returns to the reproducible factors. The longer answer qualifies this by pointing out that growth will only be positive if preferences are such that households choose high enough saving rates (and depreciation rates are not too high).

5. The equilibrium prices of the capital goods are simply their constant marginal costs B_j. The R identity implies

$$q_{j,t+1} = (r_{t+1} + \delta_j) B_j \tag{7}$$

$$= \theta_j f\left(\vec{k}_{t+1}\right) / k_{j,t+1} \tag{8}$$

The equilibrium ratio of capital inputs is given by

$$\frac{k_j}{k_{\hat{j}}} = \frac{\theta_j}{\theta_{\hat{j}}} \frac{q_{\hat{j}}}{q_j} = \frac{\theta_j}{\theta_{\hat{j}}} \frac{r + \delta_{\hat{j}}}{r + \delta_j} \frac{B_{\hat{j}}}{B_j} \tag{9}$$

Higher depreciation or marginal cost naturally reduces capital use.

6. If depreciation rates are the same, the model boils down to an AK structure because relative capital inputs are fixed by (9). Hence, $f\left(\overrightarrow{k}\right) = k_1 \Gamma$ where Γ is a constant that only depends on parameters. Similarly, total investment $\sum_j B_j i_j$ is a fixed multiple of i_1.

2 Stochastic Endowment Economy

Demographics: There is a single representative agent who lives forever. There is a representative firm (owned by the household) that also lives forever.

Preferences:

- The household maximizes $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t, \theta_t l_t)$ where $u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \theta l$. θ_t is an i.i.d. aggregate taste shock (common to all households). c is consumption and l is hours worked.

- The firm maximizes the present discounted value of profits

$$\mathbb{E} \sum_{t=0}^{\infty} D_t \pi_t \tag{10}$$

where $D_t = [R_1 \times R_2 \times ... \times R_t]^{-1}$ is a discount factor and R is the gross interest rate. $D_0 = 1$.

Endowments: The firm is endowed with $k_0 = 1$ units of land which does not depreciate.

Technology: Output can only be consumed. It is produced from labor l and land k (in fixed supply) according to

$$c_t = l_t^{\alpha} k_t^{1-\alpha} \tag{11}$$

Markets: There are competitive markets for goods (numeraire), hours (wage w), land purchases (price q), and one period discount bonds (in zero net supply; price p). The household owns the firm and receives its profits π (which it takes as given).

Questions:

1. [9 points] State the household dynamic program and derive the first-order conditions. Distinguish the individual from the aggregate state. Assume that households can only save in the form of bonds. The firm owns and trades land.
2. [3 points] Explain in words what the static optimality condition and the Euler equations say ("if the household gives up a unit of ... ").
3. [13 points] State the firm's dynamic program and derive the first-order conditions. Firms rent labor and buy and sell land. Note that land brought into period t produces output in t and can then be sold.
4. [11 points] Define a recursive competitive equilibrium.
5. [8 points] Show that the equilibrium labor input is given by $l = (\alpha/\theta)^{1/\phi}$, and that equilibrium consumption is given by $c = (\alpha/\theta)^{\frac{\alpha}{\phi}}$ where $\phi = 1 - \alpha + \alpha\sigma > 0$.
6. [8 points] Solve for the equilibrium bond price. Explain the intuition for how the price depends on θ.
7. [5 points] Consider a version of the model where the θ shocks are idiosyncratic; i.e., each household draws a different i.i.d. θ in each period. What is the aggregate state for this economy? Explain why less information would be insufficient as a state.

2.1 Solution

1. Dynamic program: The individual state is b. The aggregate state is θ. π need not be a state variable because the household knows that it is a function of θ. Therefore:

$$V(b,\theta) = \max_{c,l,b'} u(c,\theta l) + \beta \mathbb{E} V(b',\theta') \tag{12}$$

subject to the budget constraint

$$c + p(\theta) b' = w(\theta) l + b + \pi(\theta) \tag{13}$$

The first-order conditions are standard. The static condition is

$$u_c = c^{-\sigma} = u_l / w = \theta / w \tag{14}$$

and the Euler equation is

$$c^{-\sigma} = \beta \mathbb{E} (c')^{-\sigma} / p \tag{15}$$

2. In words:

   (a) Give up a unit of leisure and lose θ utils. Eat the resulting income w at marginal utility u_c.
   (b) Give up a unit of consumption and buy $1/p$ bonds.

3. Firm:

   (a) Dynamic program:

$$W(k,\theta) = \max_{l,k'} \pi(k,\theta) + R(\theta)^{-1} \mathbb{E}W(k',\theta') \tag{16}$$

   where

$$\pi(k,\theta) = l(k,\theta)^{\alpha} k^{1-\alpha} - w(\theta) l(k,\theta) - q(\theta)[k'-k] \tag{17}$$

   denotes profits and $R(\theta) = p(\theta)^{-1}$ is the return on bonds.

   (b) First-order conditions:
      i. $\alpha(k/l)^{1-\alpha} = w(\theta)$: the marginal product of labor equals the wage.
      ii. $q(\theta) = R(\theta)^{-1} \mathbb{E}W_k(k',\theta')$: equates the costs and benefits of buying capital.
   (c) Envelope: $W_k = (1-\alpha)(k/l)^{-\alpha} + q(\theta)$: land produces output and can then be sold.
   (d) Euler: $q(\theta) = R(\theta)^{-1} \mathbb{E}\left\{(1-\alpha)(k'/l')^{-\alpha} + q(\theta')\right\}$

4. RCE objects:

   (a) Household: policy funtions $c(b,\theta), l(b,\theta), b'(b,\theta)$ and a value function. Those solve the Bellman equation in the usual sense.
   (b) Firm: a value function W, a labor demand function $w(\theta) = \alpha l(k,\theta)^{\alpha-1}$, an investment function $\kappa(k,\theta)$ and a profit function $\pi(k,\theta)$. Those solve the Bellman equation in the usual sense.
   (c) Price functions $p(\theta)$, $w(\theta)$, $R(\theta)$.
   (d) Market clearing:
      i. goods: $c(0,\theta) = l(k_0,\theta)^{\alpha} k_0^{1-\alpha}$
      ii. bonds: $b'(0,\theta) = 0$
      iii. land: $\kappa(k_0,\theta) = k_0$
      iv. labor: implicit.

5. Equilibrium allocation: Since nothing is storable, we just need static optimality $c^{-\sigma} = l^{1-\alpha}\theta/\alpha$ and feasibility $c = l^{\alpha}$. Jointly, these imply

$$l = (\alpha/\theta)^{1/\phi}$$

   where $\phi = 1 - \alpha + \alpha\sigma > 0$; and therefore

$$c = (\alpha/\theta)^{\frac{\alpha}{\phi}} \tag{18}$$

6. The bond price solves the consumption Euler equation, where consumption is given by (18): Note that $u_c = (\theta/\alpha)^\rho$ where $\rho = \sigma\alpha/\phi > 0$. From the Euler equation:

$$p = \beta \mathbb{E} \left\{ (\theta'/\alpha)^\rho \right\} / (\theta/\alpha)^\rho \tag{19}$$

With the i.i.d. preference shock assumption, the expectation is simply a constant. Hence, p is increasing in θ. When θ is high, hours are low, hence consumption is low and marginal utility is high. Due to the i.i.d. assumption, this does not affect the outlook for tomorrow. Hence, savings decline and the bond price falls.

7. Heterogeneity:

   (a) In general, we need to keep track of the joint distribution of the individual states. In this case, this is the joint distribution of (b, θ). For the firm, θ disappears as a state variable.

   (b) If we knew less, e.g. only the marginal distributions of b and θ, this would not be sufficient to determine aggregate labor supply and consumption and therefore the market clearing interest rate. The model is a bit peculiar in that nobody accumulates anything, so the problem here is not that we don't know aggregate saving and hence the marginal product of capital tomorrow.

End of exam.