Final Exam. Econ720. Fall 2018

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 2 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

---

1 CIA model without capital

Demographics: A single represenative household lives forever.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t u\left(c_{1,t}, c_{2,t}, n_t\right) \tag{1}$$

There are two consumption goods, c_1 and c_2. n is hours worked.

Technology:

$$c_{1t} + c_{2t} = An_t \tag{2}$$

with $A > 0$.

Endowments: M_0 units of money in period 0. $B_0 = 0$ units of bonds in period 0. Bonds pay *nominal* interest rate R.

Government: The government prints money and hands it out as a lump-sum transfer to households.

Markets: consumption (prices $p_{1,t}, P_{2,t}$); bonds (price normalized to 1); money (numeraire); labor rental (wage $w_t p_{1,t}$).

The household's budget constraint in nominal terms is given by

$$M_{t+1} + B_{t+1} + p_{1,t}c_{1,t} + P_{2,t}c_{2,t} = M_t + w_t p_{1,t} n_t + R_t B_t + p_{1,t}\tau_{t+1} \tag{3}$$

In real terms this becomes

$$m_{t+1}\pi_{t+1} + b_{t+1}\pi_{t+1} + c_{1,t} + p_{2,t}c_{2,t} = m_t + w_t n_t + R_t b_t + \tau_{t+1} \tag{4}$$

where $m_t = M_t/p_{1,t}$, $b_t = B_t/p_{1,t}$, $p_{2,t} = P_{2,t}/p_{1,t}$, and $\pi_{t+1} = p_{1,t+1}/p_{1,t}$.

Consumption of good 1 is subject to the cash in advance constraint

$$M_t \geq p_{1,t}c_{1,t} \tag{5}$$

Questions:

1. [4 points] Write down the household's dynamic program.
2. [10 points] Derive and interpret the first-order and envelope conditions.
3. [10 points] Derive and interpret the optimality conditions $u_2/u_n = -p_2/w$, $u_1/u_2 = R/p_2$, and
$$u_2 = \beta u_2\left(.'\right)\frac{R'}{\pi'}\frac{p_2}{p_2'} \tag{6}$$
4. [4 points] For what value of the nominal interest rate does the CIA constraint not bind? Derive and explain.

5. [4 points] Define a solution to the household problem in sequence language. You should substitute out Lagrange multipliers and value function derivations.

6. [7 points] Define a competitive equilibrium.

7. [4 points] What is the welfare maximizing nominal interest rate? What is the intuition?

1.1 Answer: CIA model without capital[1]

1. Household dynamic program:

$$V(m,b) = \max u(c_1, c_{2,}n) + \lambda BC + \gamma(m-c) + \beta V(m', b') \tag{7}$$

2. First-order conditions:

$$u_1 = \lambda + \gamma \tag{8}$$
$$u_2 = \lambda p_2 \tag{9}$$
$$u_n = -\lambda w \tag{10}$$
$$\lambda \pi' = \beta V_m(.') = \beta V_b(.')$$

   Interpretation:

   (a) c_1: It takes a unit of money to buy a unit of c_1. That relaxes budget and CIA constraints.
   (b) c_2: p_2 units of income buy one unit of c_2.
   (c) n: work one hour; earn w units of income.
   (d) m', b': One unit of income buys $1/\pi'$ units of money or bonds.

   Envelope:

$$V_m = \lambda + \gamma \tag{11}$$
$$V_b = \lambda R \tag{12}$$

   Interpretation: A unit of money relaxes both constraints. One bond yields R units of income.

3. Substitute out value function derivatives:

   (a) Static condition: $u_2/u_n = p_2/w$ with standard interpretation.
   (b) Key: $V_m = V_b$ implies $\lambda + \gamma = \lambda R$. Since one unit of income buys the same amount of money and bonds, their values must be the same ($V_m = V_b$).
   (c) Therefore: $u_1/u_2 = R/p_2$. If the household takes a bond into the period, he gets $1/p_2$ units of c_2. He can also take money into the period. That gives up the nominal interest rate R.

[1] UCLA Fall 2007

(d) Euler:

$$u_2 = \beta u_2(.') \frac{R'}{\pi'} \frac{p_2}{p_2'} = \beta u_2(.') \frac{R'}{P_2'/P_2} \tag{13}$$

Giving up one unit of c_2 today allows the household to buy bonds with real interest R'/π'. With this interest, he can buy $1/p_2'$ units of c_2 tomorrow.

4. CIA constraint binds unless $\gamma = 0$ in which case $R = 1$ (the nominal interest rate is 0). The usual interpretation: holding money has no opportunity cost.

5. Solution in sequence language: $\{c_{1,t}, c_{2,t}, n_{t,} m_t, b_t\}$ that satisfy: 3 first order conditions; budget constraint; either CIA constraint or $R = 1$ in which case CIA does not bind and the household's portfolio is indeterminate. (And boundary conditions).

6. CE: household objects + $\{w_t, \pi_t, R_t, p_{2,t}\}$ that satisfy:
   (a) household (4)
   (b) firms: $w_t = A$
   (c) government: $M_{t+1} - M_t = p_{1,t}\tau_{t+1}$ or $m_{t+1}\pi_{t+1} - m_t = \tau_{t+1}$.
   (d) market clearing: goods (RC), money (implicit), labor (implicit), bonds ($b_t = 0$).

7. Optimal monetary policy: the best the government can do is make the CIA constraint not bind (Friedman Rule).

2 Continuous Time

Consider an infinitely lived agent in continuous time. Preferences are $\int_0^\infty e^{-\rho t} u(c_t)\, dt$. The budget constraint is given by

$$\dot{b}_t = rb_t + w_t l_t + \pi_t n_t - c_t \tag{14}$$

where b denotes bond holdings, r is the interest rate, w is the wage on labor l, π is profits earned from holding patents n, and c is consumption. Patents are accumulated according to

$$\dot{n}_t = \delta \bar{n}_t (1 - l_t)^\alpha \tag{15}$$

with $\delta > 0$ and $0 < \alpha < 1$. The agent takes prices (r, w, π) and $\bar{n}$ as given.

Questions:

1. [11 points] Write down the current value Hamiltonian and derive the first-order conditions.

2. [9 points] Assume that w and $\bar{n}$ grow at the same constant rate. Solve for the optimal balanced growth value of l. Explain the intuition for the result.

3. [6 points] Now consider the case $\alpha = 1$. Qualitatively, what would the household's solution for l look like? An intuitive explanation suffices.

2.1 Answer

1. Hamiltonian:
$$H = u(c) + \lambda [rb + wl + n\pi - c] + \mu [\delta \bar{n} (1 - l)^{\alpha}] \tag{16}$$
FOCs:
$$u'(c) = \lambda \tag{17}$$
$$\lambda w = \mu \delta \bar{n} \alpha (1 - l)^{\alpha - 1} \tag{18}$$
$$\dot{\lambda} = (\rho - r) \lambda \tag{19}$$
$$\dot{\mu} = \rho \mu - \pi \lambda \tag{20}$$

2. Optimal l: Constant $w/\bar{n}$ implies that
$$g(\lambda) = g(\mu) = \rho - r = \rho - \pi\lambda/\mu \tag{21}$$
Therefore, $\mu/\lambda = \pi/r$. Then
$$(1 - l)^{1-\alpha} = \frac{\delta \bar{n} \alpha}{w} \frac{\pi}{r} \tag{22}$$
Intuition: l can be used to produce 2 assets, b at marginal time cost $1/w$ and with return r, and n at marginal time cost $1/\left[\delta \bar{n} \alpha (1 - l)^{\alpha - 1}\right]$ and with return π. The FOC equates the ratios of returns to marginal costs for both assets.

3. With $\alpha = 1$, l drops out of the optimality condition. Of course, that condition would no longer be valid, unless the solution for l were interior. In general, it will not be. Unless the ratio of returns to marginal costs just happens to be equal for both assets, the household will either choose $l = 0$ or $l = 1$.

3 Asset Pricing with Habits

Demographics: An infinitely lived representative household.

Preferences: $\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t, \lambda_t \bar{c}_t)$ where

- $\bar{c}_t$ is average consumption (taken as given by the household, but $\bar{c}_t = c_t$ in equilibrium) and
- λ_t is a shock to marginal utility that follows a Markov chain.
- To make things specific, assume $u(c, \lambda_t \bar{c}_t) = \left[\frac{c_t}{\lambda_t \bar{c}_t}\right]^{1-\sigma} / (1-\sigma)$.

Endowments: There is one tree that yields a constant amount d of fruit in each period. The resource constraint is $c_t \leq d$.

Markets: There are competitive markets for goods (numeraire), trees (price p_t) and one period bonds (return R_t).

Questions:

1. [12 points] Write down the household's Bellman equation and derive the Lucas asset pricing equations for trees and bonds.
2. [4 points] Derive equilibrium risk free bond return.
3. [9 points] For the case of i.i.d. λ_t, derive the price of the stock. Explain the intuition for how p_t comoves with λ_t.
4. [6 points] Set up the planner's problem. Derive and explain why any constant $c \leq d$ is optimal.

3.1 Answers: Asset Pricing with Habits[2]

1. Bellman equation:

$$V(k, b; \lambda) = \max u([p+d]k + Rb - pk' - b', \lambda_t \bar{c}_t) + \beta \mathbb{E} V(k', b'; \lambda') \tag{23}$$

First-order conditions are standard and so are the asset pricing equations:

$$1 = \mathbb{E} MRS_{t,t+1} R'_j \tag{24}$$

where $R'_j = R'$ for the bond and $R'_j = (p' + d')/p$ for the tree.

[2]Based on Albany qualifying exam 2013.

2. Equilibrium: Obviously, $c = \bar{c} = d$. Therefore, $u'(c, \lambda\bar{c}) = \lambda^{\sigma-1}/d$. Then the $MRS_{t,t+1} = \beta(\lambda_t/\lambda_{t+1})^{1-\sigma}$. The risk free rate is given by

$$R = 1/\mathbb{E}MRS = \beta\lambda_t^{1-\sigma}\mathbb{E}\left\{\lambda_{t+1}^{\sigma-1}|\lambda_t\right\} \tag{25}$$

3. The price of the stock is, as usual, given by the discounted present value of dividends:

$$p_t = d\lambda_t^{1-\sigma}\sum_{j=1}^{\infty}\beta^j\mathbb{E}\lambda_{t+j}^{\sigma-1} \tag{26}$$

$$= d\lambda_t^{1-\sigma}\Lambda \tag{27}$$

where Λ is a constant (with i.i.d. λs). Intuition: Future marginal utilities are i.i.d. Hence, the only source of fluctuations is current marginal utility. When λ is high, marginal utility is low. Agents bid up the asset price.

4. Planner: The planner internalizes that $\bar{c}_t = c_t$. Hence, utility becomes independent of c as long as c is constant over time. The intuition is simply that the household values consumption relative to the mean, which is by construction always 1.

End of exam.