Final Exam. Econ720. Fall 2017

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 2 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Continuous time growth model

Demographics: There is a representative agent who lives forever with mass $L_t = e^{nt}$.

Preferences: $\int_0^\infty e^{-\rho t} u(c_t, 1 - l_t)\, dt$ where c is consumption per person and $1 - l$ is leisure per person. Assume that

$$u(c_t, 1 - l_t) = \frac{c_t^{1-\theta}}{1-\theta} \times h(1 - l_t) \tag{1}$$

with $h'(1 - l) > 0$.

Endowments: The agent has L_t units of time in each "period" (1 unit per capita). In $t = 0$, the agent has K_0 units of capital.

Technology:

$$\dot{K}_t + c_t L_t = F(K_t, A_t L_t l_t) - \delta K_t \tag{2}$$

where F has constant returns to scale and $A_t = e^{gt}$.

Questions:

1. [5 points] Derive the balanced growth rates of all model variables.
2. [15 points] Write down the *detrended* planner's problem. If you don't know how to detrend, write down the undetrended problem for reduced credit.
3. [8 points] Derive the necessary conditions for optimality and define a solution to the planner's problem.
4. [7 points] Show that the balanced growth path is unique if h'/h is monotone in $1 - l$.

2 Stochastic growth

Demographics: A representative household with unit mass who lives forever.

Preferences: $\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t)$.

Endowments: k_0 units of capital in $t = 0$. One unit of labor time in each period: $l_t = 1$.

Technologies: Output is produced from capital and labor; it is used for consumption and investment:

$$y_t = k_t^{\alpha} l_t^{1-\alpha} = c_t + i_t \tag{3}$$

New capital is produced from existing capital and investment:

$$k_{t+1} = k_t^{\delta_t} i_t^{1-\delta_t} \tag{4}$$

where δ_t follows a Markov chain. δ_t is a shock to something akin to depreciation.

Questions:

1. [10 points] State the planner's problem and derive necessary conditions for optimality.
2. [7 points] Derive and interpret the Euler equation

$$u'(c) = (1-\delta)(k/i)^{\delta}\beta\mathbb{E}\left\{u'(c')\left[\alpha(k')^{\alpha-1} + \frac{\delta'}{1-\delta'}(k'/i')^{-1}\right]|\delta\right\} \tag{5}$$

3. [7 points] Assuming log utility, conjecture that the consumption/output ratio depends only on δ: $c = \omega(\delta)k^{\alpha}$. Show that this conjecture satisfies the Euler equation. Hint: Note that the Euler equation may be written as

$$u'(c) = \frac{1-\delta}{i}\beta\mathbb{E}\left\{u'(c')\left[\alpha y' + \frac{\delta'}{1-\delta'}i'\right]|\delta\right\} \tag{6}$$

   and express i/c as a function of $\omega(\delta)$.

4. [5 points] If $\delta \sim i.i.d.$, does the consumption/output ratio increase or decrease in δ?
5. [12 points] Now suppose that agents differ in their initial endowments of capital. Define a Recursive Competitive Equilibrium. The market arrangements are:
   (a) A competitive final goods firm produces output according to (3).
   (b) Households accumulate capital according to (4). Capital is the only asset. It is not tradable, but can be rented to the firm. Households also work for the firm.
   (c) All markets are competitive.

   You do not need to write out the agents' problems. Just list the objects and equilibrium conditions that define an RCE.

3 Mortenson-Pissarides with Training

Consider a Mortenson-Pissarides model in continuous time.

- There are $\bar{L}$ workers who can be either unemployed (mass U), in training (mass T), or working (mass E).
- The unemployed receive zero payoff.
- Firms post vacancies subject to flow cost C. The cost is no longer paid once the firm is matched to a worker.
- Matching takes place through a constant returns to scale matching function $M(U, V)$.
- Payoffs are discounted at rate r.

Timing:

1. Firms decide on vacancies (free entry).
2. Matches form and wages are negotiated (Nash bargaining with equal bargaining weights).
3. Matched worker-firm pairs start out as trainees who produce no output. While in training, workers do not get paid.
4. With flow probability λ, a trainee turns into a worker who produces output A.
5. Matches in the work phase are destroyed with probability b.

Questions:

1. [8 points] Write down the worker's value functions.
2. [8 points] Write down the firm's value functions.
3. [8 points] Define a stationary equilibrium.

End of exam.

4 Answers

4.1 Answer: Continuous time growth model

1. Standard arguments produce $g(K) = g(C) = n + g$. Of course, l must be constant.

2. Define $\bar{k} = K/AL$ and $\bar{c} = C/A$. Then the planner solves

$$\max \int_0^\infty e^{-\rho t} e^{(1-\theta)gt} u(\bar{c}_t, 1 - l_t)\, dt \tag{7}$$

subject to

$$\dot{\bar{k}}_t = f(\bar{k}_t, l_t) - (\delta + n + g)\bar{k}_t - \bar{c}_t \tag{8}$$

The current value Hamiltonian is

$$H = u(\bar{c}, 1 - l) + \mu\left[f(\bar{k}_t, l_t) - (\delta + n + g)\bar{k}_t - \bar{c}_t\right] \tag{9}$$

3. First-order conditions:

$$\partial H/\partial \bar{c} = u_c - \mu = 0 \tag{10}$$
$$\partial H/\partial l = -u_2 + \mu f_l = 0 \tag{11}$$
$$\dot{\mu} = \hat{\rho}\mu - \mu\left[f_k - \delta - n - g\right] \tag{12}$$

where $\hat{\rho} = \rho - (1 - \theta) g$. A solution consists of functions of time $\bar{c}_t, \bar{k}_t, l_t \mu_t$ that satisfy the 3 foc's, the law of motion, $\bar{k}_0$ given, and the TVC

$$\lim_{t\to\infty} e^{\hat{\rho}t} \mu_t \bar{k}_t = 0 \tag{13}$$

From the answer, we can see why the functional form of the utility function is necessary for balanced growth. On a BGP, f_k, f_l, u_c and u_2 must be constant over time. The optimality conditions are consistent with this requirement. For any other utility function, one cannot even properly write down the detrended model.

4. BGP: Constant μ implies $f_k = \delta + n + g + \hat{\rho}$. This, in turn, implies a unique $\bar{k}/l$ and thus f_l. The resource constraint gives a unique $\bar{c}/f = 1 - (n - g - \delta)\bar{k}/f$. Finally, the first-order conditions for consumption and leisure imply

$$f_l c^{-\theta} h(1 - l) = \frac{c^{1-\theta}}{1 - \theta} h'(1 - l) \tag{14}$$

or

$$\frac{f_l}{f} \frac{h(1 - l)}{h'(1 - l)} = \frac{\bar{c}/f}{1 - \theta} \tag{15}$$

which solves for a unique l.

4.2 Answer: Stochastic growth[1]

1. Planner:
$$V(k,\delta) = \max u(k^{\alpha} - i) + \beta \mathbb{E} V\left(k^{\delta_t} i^{1-\delta_t}, \delta'\right) \tag{16}$$
FOCs:
$$u'(c) = \beta \mathbb{E} V'(.')(1-\delta)(k/i)^{\delta} \tag{17}$$
Envelope:
$$V'(k,\delta) = u'(c)\,\alpha k^{\alpha-1} + \beta \mathbb{E} V'(k',\delta')\,\delta\,(k/i)^{\delta-1} \tag{18}$$

2. Euler:
$$u'(c) = (1-\delta)(k/i)^{\delta} \beta \mathbb{E}\left\{u'(c')\,\alpha (k')^{\alpha-1} + \beta V'(k'',\delta'')\,\delta' (k'/i')^{\delta'-1}\right\} \tag{19}$$
$$= (1-\delta)(k/i)^{\delta} \beta \mathbb{E}\left\{u'(c')\left[\alpha (k')^{\alpha-1} + \frac{\delta'}{1-\delta'}(k'/i')^{-1}\right]\right\} \tag{20}$$
$$= \frac{1-\delta}{i} \beta \mathbb{E}\left\{u'(c')\left[\alpha y' + \frac{\delta'}{1-\delta'} i'\right]\right\} \tag{21}$$
where the last line above uses $k' = k^{\delta} i^{1-\delta}$ to substitute out k'. Interpretation: Giving up a unit of consumption yields $dk' = (1-\delta)(k/i)^{\delta}$. Its marginal product can be eaten next period. This is the first term in (19). In this model, having more capital also increases k'. This is represented by the second term in (19). The Euler equation lets the consumer reduce investment i' so that k'' remains unchanged and eat the resulting output.

3. The Euler equation implies
$$\frac{1-\omega(\delta)}{\omega(\delta)} = (1-\delta)\beta \mathbb{E}\left\{\frac{\alpha}{\omega(\delta')} + \frac{\delta'}{1-\delta'} \frac{1-\omega(\delta')}{\omega(\delta')} | \delta\right\} \tag{22}$$
To obtain this, simply use $i/c = (1-\omega)/\omega$. The expectation in (22) is only a function of δ. Hence, the conjecture is validated.

4. The expectation in (22) is now a constant. Higher depreciation increases ω. It is tempting to say that the intuition is that higher depreciation reduces the marginal gain from investing. However, investing one unit of consumption yields $dk' = (1-\delta)(k/i)^{\delta}$, which may or may not decline when δ is high.

5. RCE: The individual state is just k. The aggregate state is the distribution of individual k (call that S) and δ. Objects:
    (a) Household value function $V(k;S,\delta)$ and policy functions $c(k;S,\delta), i(k;S,\delta)$.
    (b) Firm: policy functions $k^f(S,\delta), l^f(S,\delta)$
    (c) Price functions $q(S,\delta), w(S,\delta)$.

[1] Based on the June 2015 preliminary exam at UC Davis.

(d) Law of motion for the aggregate state: $S' = G(S)$.

These satisfy:

(a) Households optimize in the usual sense, taking price functions and G as given.

(b) Firms optimize.

(c) Market clearing:

i. Goods: $k(S,\delta)^{\alpha} = \int [c(k;S,\delta) + i(k;S,\delta)]\, dS(k)$ where $k(S,\delta) = \int k \times dS(k)$

ii. Capital and labor rental: $k^f(S,\delta) = k(S,\delta)$ and $l^f(S,\delta) = 1$.

(d) Consistency: G is consistent with household decision rules. Define a transition function $T(B,k,S,\delta)$. This equals 1 if $k^{\delta} i(k,S,\delta) \in B$ and 0 otherwise. Then $\int T(B,k,S,\delta)\, dS(k) = G(S)(B)$. (Slight abuse of notation here.)

4.3 Answer: MP with Training

1. Workers:

$$rV_E = w + b(V_U - V_E) \tag{23}$$
$$rV_T = 0 + \lambda(V_E - V_T) \tag{24}$$
$$rV_U = 0 + a(U,V)(V_T - V_U) \tag{25}$$

where $a = M/U$ is the job finding rate.

2. Firms:

$$rW_F = A - w + b(W_V - W_F) \tag{26}$$
$$rW_T = 0 + \lambda(W_F - W_T) \tag{27}$$
$$rW_V = -C + \alpha(U,V)(W_T - W_V) \tag{28}$$

where $\alpha = M/V$ is the vacancy filling rate.

Note: We can solve for the firm's value functions in closed form. With $W_V = 0$, we have $(r+b)W_F = A - w$ and

$$(r+\lambda)W_T = \lambda W_F = \lambda \frac{A-w}{r+b} \tag{29}$$

and $W_T = C/\alpha$.

3. Stationary equilibrium: 6 value functions, E, T, U, V, w that solve:

(a) definitions of value functions above;

(b) stationarity: $bE = aT$ equates inflows into and outflows from employment. $\lambda T = a(U,V)U = M$ equates inflows into and outflows from unemployment;

(c) free entry

(d) Nash bargaining: $W_T = V_T - V_U$;

(e) identity $\bar{L} = E + T + U$.
