Final Exam. Econ720. Fall 2016

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 2 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 CIA Model with Heterogeneity (50 points)

Demographics: There is a unit measure of ex ante identical households who live forever. Time is discrete.

Preferences: $\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(1 - l_t)\right]$ where c is consumption, l denotes hours worked.

Endowments: At the beginning of time, household i is endowed with $\hat{m}_{i,0}$ units of fiat money (pieces of green paper), and $b_{i,0}$ one period real discount bonds. These are drawn from a joint distribution with density $\phi_0(\hat{m}, b)$. The total supply of money is constant over time.

Technology: $c_t = l_t$. The household operates the technology.

Markets: goods (price p), money (numeraire), bonds (pq). Bonds are issued in each period by households.

Cash-in-advance constraint: $p_t c_t \leq \hat{m}_t$.

Questions:

1. [3 points] What is the aggregate state variable for this economy?
2. [8 points] State the household's dynamic program. Hint: the budget constraint is
$$pc + pqb' + \hat{m}' = pl + \hat{m} + pb \tag{1}$$
3. [10 points] State the first-order and envelope conditions.
4. [10 points] Simplify to obtain 5 equations in c, l, m, b, μ where μ is the Lagrange multiplier on the CIA constraint. Define a solution in sequence language.
5. [5 points] Interpret the first-order conditions.
6. [14 points] Define a recursive competitive equilibrium. Hint: You will need to determine the transition function $\Pr(m' \in M, b' \in B | m, b, S)$ from the household's decision rules.

2 Asset Pricing with Habit Formation (35 points)

Demographics: A unit mass of infinitely lived, identical households.

Preferences:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t},c_{t-1}\right) \tag{2}$$

Endowments: At $t=0$, the household owns one tree.

Technology: The tree produces a random dividend that follows $d_t = G_t d_{t-1}$ with $E(G) = \bar{G} > 0$ and $G \sim iid$.

Markets: There are competitive markets for goods (numeraire), trees (p_t), and one period discount bonds (price 1; return R_t).

Questions:

1. [7 points] State the household's dynamic program. Ignore the fact that this model should be detrended first.

   Hint: It helps to define $y = k(p+d) + Rb$ as the household's cash on hand and make this a state variable.

2. [8 points] Derive first-order and envelope conditions.

3. [5 points] Simplify these conditions and define a solution in sequence language (substituting out Lagrange multipliers and value functions)

4. [4 points] Interpret the simplified first-order conditions.

5. [10 points] Assume $u(c_t, c_{t-1}) = \ln(c_t - \sigma c_{t-1})$. Show that the equilibrium risk free rate is of the form $R_t = f(G_t)$. Hint: Write u_c and $u_z(.')$ as functions of (d, G). Use the fact that $c' = Gd$ and $z = d/G$. The fact that $G \sim iid$ is key.

3 McCall Model with Promotions (15 points)

There is a representative, infinitely lived agent with preferences $\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}y_{t}$.

When unemployed, the worker

- receives $y_t = b$;
- draws a wage offer w from the distribution $F(W) = \Pr(w \le W)$. $w \in [0, B]$.

While employed, the worker

- becomes unemployed with probability δ;
- receives a promotion that raises the wage from w to θw with probability p (only if he does not become unemployed).

While promoted, the worker becomes unemployed with probability δ. On a given job, the worker can be promoted only once. Thereafter his wage remains fixed for the duration of the match.

Questions:

1. [10 points] Write down the value functions (Bellman equations) for the three worker states.
2. [5 points] How would your answer change if the worker could be promoted many times? That is, when employed, the wage rises by factor θ in each period with probability p?

End of exam.

4 Answers

4.1 Answer: CIA Model with Heterogeneity

1. The aggregate state S is the distribution of households over their states. Let the density be $\phi(m, b, S)$.

2. Bellman: The household takes as given the law of motion $S' = G(S)$.

$$V(m, b, S) = u(c, 1 - l) + \beta V(m', b', G(S)) + \lambda BC + \mu [m - c] \tag{3}$$

   where the budget constraint is

$$l + m + b - q(S) b' - \pi'(S, S') m' - c = 0 \tag{4}$$

   where $\pi' = p(S') / p(S)$.

3. FOC:

$$u_c = \lambda + \mu \tag{5}$$
$$u_l = \lambda \tag{6}$$
$$\beta V_m(') = \lambda \pi' \tag{7}$$
$$\beta V_b(') = \lambda q \tag{8}$$

   Envelope:

$$V_m = \lambda + \mu \tag{9}$$
$$V_b = \lambda \tag{10}$$

4. Simplify:

$$u_l q = \beta u_l(') \tag{11}$$
$$u_l \pi' = \beta u_c(') \tag{12}$$

5. Solution in sequence language: $\{c, l, m, b, \mu\}$ that satisfy:

   (a) 2 FOCs

   (b) budget constraint

   (c) CIA or $\mu = 0$

   (d) $u_c = u_l + \mu$

   (e) TVC

6. Interpretation:

(a) Work q extra hours, buy a bond. Cut work hours tomorrow by 1 hour.

(b) Work π' extra hours. Buy one unit of m'. Use it to eat one unit tomorrow.

(c) $u_c = u_l + \mu$: If CIA does not bind, work one hour and eat the output. If CIA does bind: eating has an additional cost μ.

7. RCE:

Objects:

(a) Household value and policy functions

(b) Price functions $p(S), q(S)$.

(c) Aggregate law of motion $G(S)$.

Equilibrium conditions:

(a) Household solves his problem in the usual sense (optimization and fixed point).

(b) Market clearing. Here, this simply requires that aggregate supply of b equals 0 and that nominal money supply, mp, is constant over time.

$$0 = \int_{m,b} \phi(m,b)\, b dm\; db \tag{13}$$

$$\bar{m}/p(S) = \int_{m,b} \phi(m,b)\, m dm\; db \tag{14}$$

(c) Consistency: Let $\Phi(M,B,S) = \Pr(m \in M, b \in B|S)$. Individual behavior introduces a transition function $\Pr(m' \in M, b' \in B|m,b,S) = F(M,B|m,b,S)$. In this model, $F(M,B|m,b,S) = 1$ if $m'(m,b,S) \in M$ and $b'(m,b,S) \in B$ and 0 otherwise. The evolution of Φ is governed by

$$\Phi(M,B,S') = \int_{m,b} \phi(m,b,S)\, F(M,B|m,b,S)\, dm\; db \tag{15}$$

The household takes as given the law of motion G. The equation above must hold when $S' = G(S)$.

4.2 Answer: Asset Pricing with Habit Formation

1. Budget constraint:

$$y = k(p+d) + Rb = c + pk' + b' \tag{16}$$

Let $z_t = c_{t-1}$. Bellman equation:

$$V(y,z,d) = \max_{c,k',b'} u(c,z) + \beta \mathbb{E} V(k'(p'+d') + R'b', c, d') \tag{17}$$

$$+ \lambda [y - c - pk' - b'] \tag{18}$$

2. FOC:

$$\beta \mathbb{E} V_x (') (p' + d') = \lambda p \tag{19}$$
$$\beta \mathbb{E} V_x (') R' = \lambda \tag{20}$$
$$u_c + \beta \mathbb{E} V_z (') = \lambda \tag{21}$$

Envelope:

$$V_x = \lambda \tag{22}$$
$$V_z = u_z \tag{23}$$

3. Simplify:

$$\lambda = u_c + \beta \mathbb{E} u_z (') \tag{24}$$
$$= \beta \mathbb{E} \left\{ \lambda' \frac{p' + d'}{p} \right\} \tag{25}$$
$$= \beta R' \mathbb{E} \lambda' \tag{26}$$

4. Solution: $\{c, k, b\}$ that satisfy 2 Lucas asset pricing equations, budget constraint, TVC.

5. Interpretation: λ is the total marginal utility of consumption. The other equations are just Lucas asset pricing equations.

6. Risk-free rate: In equilibrium, $c = d$. The key is then:

$$u_c = (d - \sigma d/G)^{-1} \tag{27}$$
$$u_z (') = -\sigma (dG' - \sigma d)^{-1} \tag{28}$$

Hence, $\mathbb{E} u_z (')$ is a function of (d, G), and so is λ. Write $\lambda = g(d, G)$. Also,

$$\mathbb{E} \lambda' = \mathbb{E} u_c (') - \sigma \mathbb{E} u_z ('') \tag{29}$$
$$= \mathbb{E} (G'd - \sigma d)^{-1} - \sigma \mathbb{E} (G'G''d - \sigma G'd)^{-1} \tag{30}$$

Since G is iid, $\mathbb{E} \lambda' = h(d, G)$. Then the FOC for bonds becomes

$$1/R' = \beta \frac{h(d, G)}{g(d, G)} \tag{31}$$

4.3 Answer: McCall Model with Promotions

1. Bellman equations:
$V_U = b + \beta Q$ where $Q = \int_0^B \max \{V_w (w), V_U\} dF(w)$.
$V_w (w) = w + \beta \delta V_U + \beta (1 - \delta) [p V_p (w) + (1 - p) V_w (w)]$
$V_p (w) = \theta w + \beta \delta V_U + \beta (1 - \delta) V_p (w)$

2. Now there only two states: working and unemployed. V_U does not change. V_w becomes $V_w(w) = w + \beta\delta V_U + \beta(1-\delta)[pV_w(\theta w) + (1-p)V_w(w)]$
