Final Exam. Econ720. Fall 2014

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 2 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Lucas Tree Model with Idiosyncratic Shocks

Demographics:

- Time is discrete and goes on forever.
- There are $j = 1, ..., J$ types of agents with mass $\mu_j = 1/J$.

Endowments:

- At $t = 0$: each person has $n_{j,0}$ shares of trees and $b_{j,0}$ units of capital.
- Trees are in fixed supply $N = \sum_j \mu_j n_{j,0}$.
- In each period, each person of type j gets a productivity draw $A_{j,t} \in \{a_1, ..., a_J\}$. Assume that in every period each a level is drawn by exactly one type j. So there is no aggregate uncertainty.

Preferences: $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_{j,t})$ where c is consumption.

Technologies:

- Trees produce dividends d_t that follow a Markov chain with transition matrix $\pi_{d,d'} = \Pr(d'|d)$.
- Each person of type j can produce output according to $y_{j,t} = A_{j,t} k_{j,t}^{\alpha}$ where k_j is capital used in production.

- Aggregate resource constraint: $Y = \sum_j \mu_j y_j$ and $Y + Nd + (1-\delta) K = C + K'$ where K is aggregate capital.

Markets:

- Goods (numeraire), capital rental (q), trees (p).
- Note that households can rent capital from others: $b_{j,t} \neq k_{j,t}$.

Questions:

1. The household's budget constraint is given by

$$c_j + qk_j + b_j' + pn_j' = A_j k_j^\alpha + (q + 1 - \delta) b_j + n_j(d + p) \tag{1}$$

   Explain why this is the correct budget constraint.

2. State the household problem as a dynamic program. What are the states and controls?
3. Derive the household's first-order conditions. Substitute out derivatives of the value functions.
4. Define a stationary Recursive Competitive Equilibrium. Assume that d is constant over time.
5. Describe a set of new assets that agents can use as insurance against the A_j shocks. How does this modify the household problem (you do not need to derive a new solution)?

   You could then show that, in equilibrium, all idiosyncratic risk is fully insured. But I am not asking you to do so.

2 Growth Model with Two Capital Goods

Demographics:

- Time is continuous and goes on forever.
- There is a single representative household who lives forever.

Preferences: $\int_0^\infty e^{-\rho t} u(C_t)$ where C is consumption and $u(C) = C^{1-\theta}/(1-\theta)$.

Endowments: At $t = 0$, the household is endowed with K_0 and H_0.

Technologies: A single good is produced according to $Y = K^\alpha H^{1-\alpha} = C + I_K + I_H$ and $\dot{K} = I_K - \delta K$ and $\dot{H} = I_H - \delta H$.

Questions:

1. State the planner's problem. For now assume that investments I_K and I_H can be negative.
2. Derive two Euler equations and define a solution to the planner's problem.
3. What can you say about the time path of K/H?
   (a) What is the intuition for this result?
4. Now consider the same model, but assume that I_H, I_K are restricted to be nonnegative and that K_0/H_0 is so high that $I_K \geq 0$ binds.

   One can show that the dynamics of this model is given by $g(C) = \frac{(1-\alpha)(K/H)^\alpha - \delta - \rho}{\theta}$ and the laws of motion for K and H (plus boundary conditions).

   (a) Define the transformed variables $\omega = K/H$ and $\chi = C/K$. Derive laws of motion for both, so we can draw a phase diagram in (ω, χ) space.
   (b) Draw a phase diagram. For simplicity, assume that the value of ω where $\dot{\omega} = 0$ in this phase ($I_K = 0$) is less than $\alpha/(1-\alpha)$, which is the value at which I_K switches on.
   (c) Discuss the dynamics of the model.

End of exam.

3 Answers

3.1 Lucas Tree Model with Idiosyncratic Shocks

1. The household comes into the period holding b, on which he earns rental price q and the undepreciated part $1 - \delta$. But in production he uses k, which he rents at price q.

2. $V(A, b, n, d) = \max u(c) + \beta \mathbb{E} V(A', b', n', d')$ subject to the budget constraint. Controls are b', n', k. I am suppressing j subscripts everywhere.

3. First order conditions yield standard Lucas equations

$$u'(c) = \beta \mathbb{E}\{u'(c') R'\} \tag{2}$$

   where $R' = q' + 1 - \delta$ for b and $R' = (p' + d')/p$ for the tree. Then there is a static condition $q = \alpha A k^{\alpha-1}$.

4. Stationary RCE:

   (a) aggregate state: joint distribution of households over states. In this case, this is simply $S_j = (\bar{n}_j, \bar{b}_j, A_j)$ stacked for all j.

   (b) objects:

      i. V and policy functions for n, b, k. Note that there is no aggregate uncertainty, so we don't have to explicitly keep track of the distribution of households in the value functions and policy functions.

      ii. price functions $p(S), q(S)$

   (c) equilibrium conditions:

      i. household: standard

      ii. market clearing: $\sum_j \mu_j \bar{n}_j = N$, $\sum_j \mu_j \bar{b}_j = \sum_j \mu_j k_j$, goods (resource constraint).

      iii. stationarity: Law of motion for S generates a stationary distribution. But here we run into the next point...

   (d) A detail: a stationary RCE generically does not exist for finite J. One might think that agents are simply moving around among the elements of the S_j, but that cannot work. Individuals with given A_j today have different histories and therefore different choices compared with individuals in state A_j yesterday. So we need a law of large numbers to get stationarity. The stationarity condition is then standard.

5. Arrow securities: Index possible aggregate states by z. Each z corresponds to one particular vector $(A_1, ..., A_J)$. Create a set of assets that pay 1 unit of consumption next period iff any given state z' is realized.

   Household problem: add $x(z_{1,...,} z_Z)$ to the state vector and add $x'(z_{1,...,} z_Z)$ to the controls. Add $x(z)$ to income and add $\sum_z p(z'|z) x'(z')$ to spending.

3.2 Answer: Growth Model with Two Capital Goods[1]

1. Planner: The best way of setting this up is with one state variable: $Z = K + H$ with $Y = Zz^{\alpha}(1-z)^{1-\alpha}$ and $\dot{Z} = I - \delta Z$. Then

$$J = u\left(Zz^{\alpha}(1-z)^{1-\alpha} - I\right) + \mu(I - \delta Z) \tag{3}$$

With some abuse you can also set up

$$J = u\left(K^{\alpha}H^{1-\alpha} - I_K - I_H\right) + \nu(I_K - \delta K) + \mu(I_H - \delta h) \tag{4}$$

2. First-order conditions:

$$u'(C)\,Z\alpha z^{\alpha-1} = u'(C)\,Z(1-\alpha)(1-z)^{-\alpha} \tag{5}$$
$$u'(C) = \mu \tag{6}$$
$$\dot{\mu} = \rho\mu - \partial J/\partial Z \tag{7}$$

Euler:

$$g(C) = \frac{\alpha(K/H)^{\alpha-1} - \delta - \rho}{\theta} = \frac{(1-\alpha)(K/H)^{\alpha} - \delta - \rho}{\theta} \tag{8}$$

Solution: C_t, K_t, H_t that solve: 2 Euler equation, resource constraint, initial conditions, transversality.

1. K/H must be constant over time at $K/H = \alpha/(1-\alpha)$.

   (a) Intuition: The planner can exchange K for H one-for-one. Therefore, the ratio of marginal products must be fixed. If the planner starts out with the "wrong" K/H, he simply relabels some K and H or vice versa. This is really an AK model.

2. Binding non-negativity:

   (a) Start from $g(\omega) = g(K) - g(H)$. $g(K) = -\delta$. $g(H) = \omega^{\alpha} - \delta - \chi\omega$. Then

$$g(\chi) = g(C) + \delta = \frac{(1-\alpha)\omega^{\alpha} - \delta - \rho}{\theta} + \delta \tag{9}$$

   (b) $\dot{\omega} = 0$ implies $\chi = \omega^{\alpha-1}$, which is downward sloping. $\dot{\chi} = 0$ implies a fixed $\tilde{\omega}$.

   (c) Dynamics: The phase diagram reveals that χ rises monotonically and ω falls monotically. This is what we expected. The planner lets K depreciate until he reaches the one $\omega = K/H$ where investment in both K and H are positive. So ω falls over time. Rising C/K is expected because K is falling faster than output (the planner still invests in H).

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[1] Based on Barro and Sala-i-Martin, Economic Growth, Appendix 5A.