Final Exam. Econ720. Fall 2010

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- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 2:00 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Stochastic CIA Model

Consider the following stochastic growth model with a cash-in-advance constraint.

Demographics: Time is discrete. There is a representative household who lives forever. He has unit mass.

Preferences:

$$E \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(n_t\right) \right] \tag{1}$$

where $0 < \beta < 1$, c is consumption, n is labor supply, u is strictly concave with $u'(0) = \infty$ and $u'(\infty) = 0$, v is strictly convex with $v'(0) = 0$ and $v'(1) = \infty$.

Technology: A single, nonstorable good is produced according to

$$y_t = \gamma_t n_t \tag{2}$$

where γ_t is a random productivity shock with transition matrix $\Pi(\gamma', \gamma)$.

Resource constraint: The good is not storable, so that $c_t = y_t$.

Government: The government distributes money as lump-sum transfers (τ_t):

$$\bar{M}_{t+1} = \bar{M}_t + P_t \tau_t = \theta_t \bar{M}_t \tag{3}$$

$\bar{M}_t$ is the aggregate stock of money balances. θ_t is the random money growth rate.

Timing during period t:

1. Enter the period holding M_t and B_t units of a nominal bond that pays one unit of money in t.

2. Learn θ_t and γ_t.

3. Receive money transfers $P_t\tau_t$.

4. Trade bonds for money. The bond price is S_t.

5. Work and produce.

6. Buy consumption goods with cash.

7. Receive labor earnings, $P_t w_t n_t$, from the firm.

The resulting budget constraint for the household is given by

$$S_t B_{t+1} + M_{t+1} + P_t c_t = M_t + B_t + P_t\tau_t + P_t w_t n_t \tag{4}$$

and the CIA constraint is

$$P_t c_t \leq M_t + P_t\tau_t + B_t - S_t B_{t+1} \tag{5}$$

P and $\bar{M}$ grow over time. In preparation for setting up the Bellman equation, make the budget constraint and the CIA constraint stationary by dividing through by $\bar{M}_t$. Define lower case variables as $b_t = B_t/\bar{M}_t$, $p_t = P_t/\bar{M}_t$, etc. Let $m_t = M_t/\bar{M}_t$. Then

$$\begin{aligned} S_t b_{t+1}\theta_t + m_{t+1}\theta_t + p_t c_t &= m_t + b_t + p_t\tau_t + p_t w_t n_t & (6) \\ p_t c_t &\leq m_t + p_t\tau_t + b_t + S_t b_{t+1}\theta_t & (7) \end{aligned}$$

Questions:

1. Write down the Bellman equation (value when θ and γ are known) and take first-order conditions / envelope conditions.

2. What is the nominal interest rate in this economy? Show that the CIA constraint binds if the nominal interest rate is positive.

3. Simplify the first-order conditions by eliminating Lagrange multipliers and derivatives of the value function. Show that the Lucas asset pricing equation holds for the bond. Hint: Use the fact that the real bond return is given by $R' = P/(P'S)$ and $P/P' = p/(p'\theta)$.

4. Define a competitive equilibrium. For notational convenience, just write: "stochastic processes for these N variables $\{...\}$ that satisfy the following N equations." (That's shorter than writing everything as a function of histories.)

5. From hereon assume that the CIA constraint binds. Show

$$\begin{aligned} p_t \gamma_t n_t &= \theta_t \\ n_t v'(n_t) &= \beta E_t \left[\frac{\gamma_{t+1} n_{t+1} u'(\gamma_{t+1} n_{t+1})}{\theta_{t+1}} \right] \end{aligned} \tag{8}$$

which is the stochastic law of motion for hours worked. Hint: For the first equation, stare at the CIA constraint and impose equilibrium conditions to get rid off objects such as b.

6. If the shocks θ_t, γ_t are i.i.d., (8) implies that n_t is constant over time (not obvious, but assume this is true). This implies, surprisingly, that the current money shock θ_t has no real effects (it does not affect c, n, y). Does this also imply that monetary policy (the distribution of θ_t) is irrelevant? What is the intuition for this strange finding?

2 Search

Demographics: There is a unit measure of ex ante identical agents. All live forever.

Preferences: Agents value consumption c_t and dislike work effort a_t:

$$E_0 \sum_{t=0}^{\infty} R^{-t}[u(c_t) - a_t] \tag{9}$$

where u is strictly concave with $u(0) = 0$.

The timing of events:

- Agents are either "employed" (mass E) or "unemployed" (mass $U = 1 - E$).
- An unemployed agent consumes $c = 0$.
- When unemployed: an agent can produce y units of the consumption good. The effort cost is $a = \alpha_i$ with probability ρ_i. $i \in \{1, 2\}$. The agent knows a before choosing whether to produce.
- If an unemployed agent chooses to produce, he becomes employed in the next period.
- An employed agent meets a trading partner with probability bE_t. The pair then trades and consumes $c = y$. In the next period, both partners become unemployed.
- If an employed agent does not meet a partner, he consumes 0 and remains employed.

Questions:

1. Write down the Bellman equation for an employed person.
2. Write down the Bellman equation for an unemployed person. What objects does he choose (think mixed strategies).
3. State the laws of motion for the aggregate state variables E_t and U_t.
4. Do you think the steady state of the economy is unique? Explain why or why not.

3 Answers

3.1 Answer: Stochastic CIA Model

Based on a problem due to Steve Williamson.

1. Bellman equation. The value after the shocks for t are realized is given by

$$\begin{aligned} V(m, b, \theta, \gamma) &= \max u(c) - v(n) + \beta EV(m', b', \theta', \gamma') \quad (10) \\ &\quad +\lambda[CIA] + \mu[BC] \quad (11) \end{aligned}$$

First order conditions:

$$\begin{aligned} c &: u'(c) = (\lambda + \mu) p \quad (12) \\ n &: v'(n) = \mu p w \quad (13) \\ m' &: \beta EV_m(.') = \mu\theta \quad (14) \\ b' &: \beta EV_b(.') = (\lambda + \mu) S\theta \quad (15) \end{aligned}$$

Envelope:

$$V_m = V_b = \lambda + \mu \quad (16)$$

Substitute envelope conditions to eliminate derivatives of V:

$$\begin{aligned} \mu\theta &= \beta E(\lambda' + \mu') \quad (17) \\ &= (\lambda + \mu) S\theta \quad (18) \end{aligned}$$

2. Nominal interest rate. The nominal interest rate is the return on the nominal bond: $1/S - 1$. The nominal interest rate is positive if $S < 1$. The CIA constraint binds when $\lambda > 0$. $S = \mu/(\lambda + \mu)$.

3. Simplify first-order conditions.

$$\begin{aligned} u'(c) Sw &= v'(n) \quad (19) \\ S\theta \frac{u'(c)}{p} &= \beta E \frac{u'(c')}{p'} \quad (20) \end{aligned}$$

These look strange because $p = P/M$. Replacing p's yields:

$$u'(c) = \beta E \frac{u'(c') P}{P'} \frac{1}{S} \quad (21)$$

which is the standard pricing formula: The real interest rate on the bond is $R' = P/(P'S)$.

4. Equilibrium: Objects: stochastic processes of c, n, b, m, y, p, S, w.

Equilibrium conditions:

- Household: 2 FOCs, budget constraint, CIA constraint (or $S = 1$).
- Firm: First-order condition $w = \gamma$ and production function $y = \gamma n$.
- Government budget constraint: gone, already imposed on the household's constraints.
- Goods market clearing: $c = y$.
- Bond market clearing: $b = 0$.
- Money market clearing: $m = 1$.

5. Characterization. The CIA constraint with $b = 0$, $c = \gamma n$, and $p\tau = \theta - 1$ implies $p\gamma n = \theta$. For the 2nd equation, start from the FOC

$$\begin{aligned} u'(c)\, Sw &= v'(n) \\ \beta E \frac{u'(c')\, p}{p'} \frac{w}{\theta} &= v'(n) \end{aligned}$$

Apply $c' = \gamma' n'$ and $w'\gamma$ and $p = \theta / (\gamma n)$ to have

$$v'(n) = \beta E \frac{u'(\gamma' n')\, \theta \gamma' n'}{\gamma n \theta'} \frac{\gamma}{\theta}$$

and cancel terms.

6. iid shocks. The distribution of θ (monetary policy rule) still matters. While the LHS of (8) is a constant, it does depend on the distribution of θ. What matters about monetary policy here is anticipated money growth, which affects the real return on money and thus the "tightness" of the CIA constraint. With iid shocks, expected money growth is independent of θ.

7. Additional issues:

1. Solve the Social Planner's problem for the optimal n_t^*. Assume that $\gamma_t = \gamma$ for all t.
2. From (8), show that a constant money growth rate (θ^*) achieves the first best. Derive that growth rate. Show that is obeys the Friedman Rule.

3.2 Answer: Search

Based on a question due to Steve Williamson.

1. Employed: The employed person makes no choices.

$$V_E = bE_t[u(y) + \beta V_U] + (1 - bE_t)[0 + \beta V_E] \tag{22}$$

2. Unemployed: The unemployed chooses the probability of producing when the cost is high and when it is low. Call those π_1 and π_2.

$$V_U = \sum Pr(a = \alpha_i) \max_{\pi_i} 0 + \beta\left[\pi_i(V_E - \alpha_i) + (1 - \pi_i)V_U\right] \tag{23}$$

Note: Generically, the agent will choose a corner solution, unless $V_E - \alpha_i = V_U$.

3. State variables: Fraction $\Pi = \sum Pr(a = \alpha_i)\pi_i$ of the unemployed chooses to produce and adds to E_{t+1}. Fraction $1 - bE_t$ of the employed does not find a trading partner and stays employed.

$$E_{t+1} = (1 - E_t)\Pi + (1 - bE_t)E_t \tag{24}$$

Fraction $1 - \Pi$ of the unemployed does not produce and remains unemployed. Fraction bE_t of the employed trades and becomes unemployed next period:

$$U_{t+1} = (1 - \Pi)U_t + b(1 - U_t)U_t \tag{25}$$

4. Steady states: There are generically multiple steady states. Notably:

1. $\pi_i = 0$ implies $E = bE = 0$ and $V_U = V_E = 0$. That makes $\pi_i = 0$ (never produce) the optimal strategy.
2. Another possible steady state has $V_E - V_U = \alpha_i$ for some i. Then π_i is indeterminate and can (subject to parameter restrictions) be chosen to satisfy the steady state law of motion (25).

Intuition: the usual market thickness externality. If more people produce they raise the probability of finding trade partners, which makes it more profitable to produce.

End of exam.