Macroeconomics Qualifying Examination

June 2021

Department of Economics

UNC Chapel Hill

Instructions:

- This examination consists of **4** questions. Answer all questions.
- The total time is 3 hours. The total number of points is 180.
- If you believe a question is ambiguously stated, ask for clarification. Unnecessary simplifications will be penalized.
- **Write legibly**.
- Number your answers.
- Explain your answers.
- Budget your time wisely. Don't get hung up on one question.
- Good luck!

Special instructions for remotely administered exams:

- This exam is open book. You may consult books and notes.
- To ask questions during the exam: call, text, or facetime Lutz Hendricks at 919-886-6885.

1 Learning by doing

This problem adds learning by doing to a standard real business cycle model. The economy's production function is

$$Y_t = z_t K_t^{\alpha} H_t^{1-\alpha},$$

where Y_t is output, z_t is a standard (Markov) technology shock, K_t is capital, and H_t are *efficiency units* of labor (defined below).

The representative household has preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) - \frac{n_t^2}{2} \right]$$

where c_t denotes the representative household's consumption in period t, and n_t denotes the representative household's hours worked in period t. The representative household's efficiency units of labor in period t are equal to $x_t n_t$, where x_t is the household's skill level. The representative household's skill level evolves according to

$$x_{t+1} = n_t^{\theta}$$

where $\theta \geq 0$ is a constant. In other words, the household's future skill depends on hours worked today. As usual, the representative household owns capital, which accumulates through investment and depreciates at rate δ.

1. (15 points) Write down and solve the social planner's problem for this economy. You may do so using either a Lagrangian or a recursive formulation.

   You should obtain the following optimality condition for consumption:

   $$\frac{1}{C_t} = \beta \mathbb{E}_t \frac{1}{C_{t+1}} \left[1 - \delta + \alpha z_{t+1} K_{t+1}^{\alpha-1} X_{t+1}^{1-\alpha} N_{t+1}^{1-\alpha} \right] \tag{1}$$

   and the following optimality condition for hours worked:

   $$N_t = (1-\alpha) \frac{Y_t}{N_t} \frac{1}{C_t} + \beta\theta (1-\alpha) \mathbb{E}_t \frac{Y_{t+1}}{N_t} \frac{1}{C_{t+1}} \tag{2}$$

   (If you got stuck on the derivation, feel free to use (1) and (2) in your answers below.)

   Answer

   We will do this recursively. The social planner's problem is

   $$V(X, K, z) = \max_{C,N,K'} \ln(C) - \frac{N^2}{2} + \beta \mathbb{E} V\left(N^{\theta}, K', z'\right)$$

   subject to

   $$C + K' = (1-\delta) K + z K^{\alpha} X^{1-\alpha} N^{1-\alpha}$$

Letting λ be the Lagrange multiplier, we get the first-order conditions for C, N, and K', respectively, as

$$\frac{1}{C} = \lambda$$
$$N = (1-\alpha)\,\lambda z K^{\alpha} X^{1-\alpha} N^{-\alpha} + \beta\theta N^{\theta-1}\mathbb{E}V_1\left(N^{\theta}, K', z'\right)$$
$$\lambda = \beta\mathbb{E}V_2\left(N^{\theta}, K', z'\right)$$

We also have the envelope conditions

$$V_1(X, K, z) = (1-\alpha)\,\lambda z K^{\alpha} X^{-\alpha} N^{1-\alpha}$$

and

$$V_2(X, K, z) = \lambda\left[1 - \delta + \alpha z K^{\alpha-1} X^{1-\alpha} N^{1-\alpha}\right]$$

Combining the first-order conditions for C and K' with the envelope condition for V_2, we get (1). Combining the first-order conditions for C and N with the envelope condition for V_1, we get

$$N = (1-\alpha)\, z K^{\alpha} X^{1-\alpha} N^{-\alpha}\frac{1}{C} + \beta\theta(1-\alpha)\, N^{\theta-1}\mathbb{E}z'\left(K'\right)^{\alpha}\left(X'\right)^{-\alpha}\left(N'\right)^{1-\alpha}\frac{1}{C'}$$

But we know $X' = N^{\theta}$, which means $N^{\theta-1}\left(X'\right)^{-\alpha} = N^{\theta-1-\theta\alpha} = N^{\theta(1-\alpha)-1}$. Then, using the production function gives (2).

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2. (5 points) Interpret the optimality condition (2) for hours worked and explain how it differs from the standard real business cycle model.

Answer

As usual, this condition states that the marginal cost of an additional hour equals the marginal benefit. What is different is that now the marginal benefit includes the effect of current hours on future productivity, hence the second term in (2), which would be absent if $\theta = 0$.

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3. (5 points) Discuss how (if at all) the addition of learning by doing might help the model overcome some of the empirical shortcomings of the standard real business cycle model. Refer to the above equations if possible.

Answer

Learning by doing may provide an additional propagation mechanism for productivity shocks, which has been a challenge for the standard RBC model. When there is an increase in z_t, this induces an increase in current hours, but this in turn raises future skill levels, which incentivizes higher hours in the future, and so on. Thus, a transitory shock to z_t may have a persistent effect on output.

Now, suppose that the law of motion for the representative household's skill level is

$$x_{t+1} = \left(n_t^{\mu} \mathcal{N}_t^{1-\mu}\right)^{\theta}, \quad \theta \geq 0, \mu \in [0,1]$$

where $n_t =$ the household's hours worked, and $\mathcal{N}_t =$ the **aggregate** hours worked.

4. (20 points) Define a recursive competitive equilibrium for this economy. Very clearly state what are the aggregate and individual state variables.

   (Think of the market wage as the wage per efficiency unit of labor. In other words, if a household's efficiency units of labor are xn, then its labor income is wxn, where w is competitively determined.)

   Answer ___

   The individual states are x, k. The aggregate states are X, K, z (NOT N). A RCE consists of

   - A value function $V(x, k, X, K, z)$ together with individual decision rules

   $$c(x,k,X,K,z), n(x,k,X,K,z), x'(x,k,X,K,z), k'(x,k,X,K,z)$$

   - Aggregate decision rules

   $$C(X,K,z), N(X,K,z), X'(X,K,z), K'(X,K,z)$$

   - Pricing functions $w(X,K,z)$, $r(X,K,z)$

   such that:

   - The value function together with individual decision rules solves

   $$V(x,k,X,K,z) = \max \ln c - \frac{n^2}{2} + \beta \mathbb{E} V(x', k', X', K', z')$$

   subject to

   $$c + k' = w(X,K,z)\, xn + (1 - \delta + r(X,K,z))\, k,$$
   $$x' = \left(n^{\mu} N(X,K,z)^{1-\mu}\right)^{\theta}$$

   and taking as given the aggregate decision rules.

   - The pricing functions satisfy

   $$w(X,K,z) = (1-\alpha)\, zK^{\alpha} (XN(X,K,z))^{-\alpha}$$
   $$r(X,K,z) = \alpha z K^{\alpha-1} (XN(X,K,z))^{1-\alpha}$$

- Aggregate consistency holds.

$$C(X,K,z) = c(X,K,X,K,z)$$

$$N(X,K,z) = n(X,K,X,K,z)$$
$$X'(X,K,z) = x'(X,K,X,K,z)$$
$$K'(X,K,z) = k'(X,K,X,K,z)$$

- Markets clear.

$$C(X,K,z) + K'(X,K,z) = (1-\delta)K + zK^{\alpha}(XN(X,K,z))^{1-\alpha}$$

and (though this is implied by earlier conditions)

$$X'(X,K,z) = N(X,K,z)^{\theta}$$

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2 Self-insurance and redistribution

Consider an economy with a large measure of infinitely-lived households, whose utility is $\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t)$, where u satisfies $u'(c) > 0$, $u''(c) < 0$ and $\lim_{c\to 0} u'(c) = \infty$.

Each household's **idiosyncratic** income y is drawn each period from $\{y_1, y_2, ..., y_N\}$, where $0 < y_1 < y_2 < ... < y_N$, and follows a Markov process. The transition matrix for y satisfies $\pi_{ij} > 0$ for all i, j, where $\pi_{ij} = Prob(y' = y_j | y = y_i)$.

Households cannot trade state-contingent claims but can borrow and save at an exogenous, fixed interest rate r. There are no aggregate shocks.

1. (5 points) Write down a natural borrowing constraint for the household. Does it depend on its current income? Also, please explain (i) why the model needs a borrowing constraint, and (ii) if the natural borrowing constraint would ever bind.

Answer

The natural borrowing constraint prohibits the household from borrowing more than it could repay in every state without setting its consumption to zero, hence

$$a_{t+1} \geq -\frac{y_1}{r}$$

This is independent of current income, since y_1 is reachable with positive probability from any current income. The model needs a borrowing constraint to rule out the possibility of perpetual borrowing (a Ponzi scheme) that would make the household's problem unbounded. If the natural borrowing limit is imposed, it will not bind at the optimum, because of the Inada condition on the utility function.

For the remainder of this problem, suppose that households can save but cannot borrow.

2. (10 points) Write down the household's recursive problem. What are its aggregate and individual state variables? What if $\pi_{ij} = \Pi_j$ is independently of i?

Answer

$$V(a, y) = \max_{c, a'} u(c) + \beta \sum_{y'} \pi(y'|y) V(a', y')$$

subject to $c \geq 0$, $a' \geq 0$, and

$$c + a' = y + (1 + r) a$$

The individual states are a and y. There are no aggregate state variables, since prices (in this case the interest rate) are exogenously given.

If $\pi_{ij} = \Pi_j$ is independently of i, we can write the household's problem with only *one* individual state variable, namely $x = y + (1+r)\,a$.

$$V(x) = \max_{c,a'} u(c) + \beta \sum_{y'} \Pi(y')\, V(y' + (1+r)\, a')$$

subject to $c \geq 0$, $a' \geq 0$, and

$$c + a' = x$$

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3. (5 points) True or false? "If the borrowing constraint binds for a household with a particular y, then it also binds for any household with a lower y."

   You do not have to provide a formal proof, but briefly explain/justify your answer.

   Answer

   This is wrong for two distinct reasons. First, the optimal borrowing amount depends on expectations of future income. Second, even if y is i.i.d. (in which case the first channel is moot), optimal borrowing amount depends on cash-on-hand, not just income. In simple terms, a household with lower y might have higher assets, so we cannot determine which household would want to borrow more.

Suppose that there is a government in this economy. It taxes the interest income of all its households at a proportional rate τ and redistributes the tax revenues as lump-sum transfers. Other than this, maintain all the previous assumptions.

We consider a recursive competitive equilibrium (not necessarily a stationary one).

4. (5 points) Write down the government's budget constraint. Clearly define your notation.

   Answer

   $$T = \tau r \int_{A \times Y} a \Phi(da, dy)$$

   where T is the lump-sum transfer, and Φ is the distribution of households across wealth and income states.

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5. (10 points) Write down the household's recursive problem. What are its aggregate and individual state variables? Very carefully explain how and why your answer differs from part 2 above.

Answer

$$V(a, y, \Phi) = \max_{c, a'} u(c) + \beta \sum \pi(y'|y) V(a', y', \Phi')$$

subject to $c \geq 0$, $a' \geq 0$, and

$$c + a' = y + (1 + r(1 - \tau)) a + T(\Phi)$$

$$\Phi' = \mathcal{H}(\Phi)$$

In this case, the distribution of households enters as an aggregate state variable, because the lump-sum transfer is a function of this distribution.

3 Two Lucas Trees

Demographics: There is a single representative household who lives forever.

Endowments:

- There are $N = 2$ types of fruit trees of unit mass ($K_j = 1; j = 1, \ldots, N$). Trees stay around forever.
- In period t, each tree produces $d_{j,t}$ fruits of type j.
- $d_{j,t}$ follows a Markov chain.
- **Note**: Each tree produces a different kind of fruit, not just a different quantity.

Preferences:

$$\mathbb{E} \sum_{t=1}^{\infty} \beta^t u\left(\alpha_{1,t} c_{1,t}, \alpha_{2,t} c_{2,t}\right) \tag{3}$$

where $\alpha_{1,t} \equiv 1$ and $\alpha_{2,t}$ alternates between $\bar{\alpha}$ in even periods and $\alpha_{2,t} = 0$ in odd periods.

Technologies: Fruit can only be eaten in the period in which it ripens: $c_{j,t} = d_{j,t}$.

Markets: There are competitive markets for fruit (prices $q_{j,t}$), trees (prices $p_{j,t}$), and riskless bonds (in zero net supply; interest rate r_t). Fruit 1 is the numeraire, but it is easier to keep $q_{1,t}$ in the model.

Questions

1. [10 points] Write out the household's dynamic program. Be careful about the budget constraint.

   Answer ______________________________

   (Based on a qualifying exam at Penn in 2019) It is easiest to write everything with prices $q_{j,t}$ and preference weights $\alpha_{j,t}$.

   - The aggregate state is $S = (d_1, d_2, \alpha_1, \alpha_2)$.
   - The indivdual state is b and $(k_j)_{\forall j}$.
   - Bellman equation:

   $$V\left(k_1, \ldots, k_N, b; S\right) = \max_{k'_j, b', c_j} u\left(\alpha_1 c_1, \alpha_2 c_2\right) + \beta \mathbb{E} V\left(k'_1, \ldots, k'_N, b'; d'_1, \alpha'_1 \alpha'_2\right) \tag{4}$$

   subject to the budget constraint

   $$\sum_j q_j c_j + \sum_j p_j k'_j + b' = \sum_j \left(p_j + q_j d_j\right) k_j + (1 + r)\, b \tag{5}$$

   and the law of motion of α. In words: Holding a type 2 tree yields the tree's price p_2 (in units of account) plus d_2 units of goods 2, valued at q_2.

2. [15 points] Derive the Lucas asset pricing conditions $\mathbb{E}\left\{\frac{\beta u_1\left(\alpha_1' c_1', \alpha_2' c_2'\right)}{u_1(\alpha_1 c_1, \alpha_2 c_2)} R'\right\} = 1$ where R is the gross return on each asset. Also derive a static condition that governs the consumption of both goods within a period.

Answer

- $\alpha_j u_j = \lambda q_j$ (which implies that $q_2 = 0$ when $\alpha_2 = 0$).
- $\beta \mathbb{E} V_j(.') = \lambda p_j$
- $\beta \mathbb{E} V_b(.') = \lambda$
- $V_j(.) = \lambda(p_j + q_j d_j)$
- $V_b(.) = \lambda(1+r)$

Combine to obtain the Lucas equations of the form

$$\alpha_j u_j = \beta \mathbb{E} \alpha_j' u_j(.') \frac{p_j' + q_j' d_j'}{p_j} \tag{6}$$

For tree 1 this is the standard equation with $\alpha_1 \equiv q_1 \equiv 1$. But for tree 2, the solution cycles between odd and even periods. In periods when $\alpha_2 = 0$, we have $q_2 = 0$ and $c_2 = 0$. For periods where both goods are valued, we have a static condition $\alpha_2 u_2 / (\alpha_1 u_1) = q_2 / q_1$.

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3. [10 points] Define a Competitive Equilibrium in recursive language.

Answer

- Value function V and policy functions that give c_j, k_j' and b' as functions of $(k_1, \ldots, k_N, b; S)$. These solve the household problem in the usual sense.
- Price functions: $p_j(S), r(S), q(S)$.
- Market clearing:
  - Goods: $c_j(s, S) = d_j$ (except when $\alpha_j = 0$)
  - Trees: $k_j'(s, S) = 1$.
  - Bonds: $b'(s, S) = 0$.

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4. [10 points] Consider the option to buy tree 2 tomorrow at the locked-in price $\hat{p}$. For a period with $\alpha' = 0$ (no dividend tomorrow), and assuming that the strike price $\hat{p}$ is always less than p_2', derive the Lucas asset pricing equation

$$1 = \mathbb{E} \frac{\beta u_1'}{u_1} \frac{\hat{p}}{p_2 - \pi} \tag{7}$$

where π is the price of the option today. Provide the economic intuition for this expression. Hint: Think of the alternatives of buying the stock today versus buying the option today.

Answer

The value of the option tomorrow is $\max(0, p_2' - \hat{p})$. So the Lucas asset pricing equation is

$$\pi u_1 = \beta \mathbb{E} u_1' \times \max(0, p_2' - \hat{p})$$

If the option is always in the money:

$$\pi u_1 = \beta \mathbb{E}\{u_1' p_2'\} - \beta \mathbb{E}\{u_1'\} \hat{p} \tag{8}$$

In a period where $\alpha' = 0$ we have $p_2 u_1 = \beta \mathbb{E} u_1' p_2'$. Therefore

$$(p_2 - \pi) u_1 = \beta \mathbb{E} u_1' \hat{p} \tag{9}$$

Interpretation: Compared with buying the stock directly, today the household saves $p_2 - \pi$, but tomorrow he has to pay $\hat{p}$ to own the stock.

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5. [10 points] Now consider a version of the model where half of the agents have $\alpha_{2,t} = \bar{\alpha}$ in all periods while the other half alternate between $\bar{\alpha}$ and $\alpha_{2,t} = 0$ as before. What would the trade pattern in this economy look like in even and odd periods. Support your answer with equations. For simplicity, assume that utility is $u(c_{1,t}, \alpha_{2,t} c_{2,t}) = v(c_{1,t}) + v(\alpha_{2,t} c_{2,t})$ with $v(c) = c^\phi / \phi$.

Answer

Both agents share the same consumption growth rate of c_1; this follows from the Euler equation. Both also share the same two period growth rate for c_2; again from the Euler equation.

It is tempting to conclude that agents who don't value good 2 next period should not buy tree 2 this period. But that conclusion is wrong. Both agents should hold the same portfolios in all periods. This follows from the Euler equation for good 1. The rate of return of the assets does not depend on preferences. The agents trade goods, not assets.

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4 Taxing Capital, Labor, and Consumption

Demographics: There is a representative household who lives forever.

Preferences: $\sum_{t=1}^{\infty} \beta^t [U(c_t) + V(1-l_t) + G(g_t)]$ where all sub-utility functions are well behaved.

Technology: $c_t + g_t + k_{t+1} = F(k_t, l_t) + (1-\delta) k_t$ where F has constant returns to scale and is well behaved.

Endowments: Households are endowed with k_0 at $t = 0$.

Markets: There are competitive markets for goods (numeraire), labor (w_t), and capital rental (q_t). Firms are simple and don't own anything.

Government: The government taxes

- labor income at rate $\tau_{l,t}$;
- consumption at rate $\tau_{c,t}$;
- capital income at rate $\tau_{k,t}$.

The government budget is balanced: $\tau_{l,t} w_t l_t + \tau_{c,t} c_t + \tau_{k,t} q_t k_t = g_t$.

Notation: c: consumption; l: hours worked; g: government consumption. It helps to define $\lambda_c \equiv 1 + \tau_c$; $\lambda_l \equiv 1 - \tau_l$; $\lambda_k \equiv 1 - \tau_k$.

Questions:

1. [5 points] Write down the household's dynamic program.

   Answer

   - Budget constraint: $k' + \lambda_c c = \lambda_l w l + \lambda_q q k$.
   - Bellman: $V(k; S) = \max U(c) + \beta V(\lambda_l w l + \lambda_q q k - \lambda_c c; S')$ where $S = (K, \tau_c, \tau_l, \tau_k, g)$.

2. [10 points] Define a solution in sequence language. Interpret the first-order conditions.

   Answer

   FOC:

   - static: $\frac{U'(c)}{V'(1-l)} = \frac{\lambda_c}{\lambda_l w}$. This is standard. Giving up one unit of consumption (which costs λ_c) allows the household to buy $\lambda_l w / \lambda_c$ units of leisure.
   - Euler: $U'(c) = \beta U'(c') \lambda_k' q' \frac{\lambda_c}{\lambda_c'}$. This is also standard, except for the intertemporal distortion coming from time varying consumption taxes.

3. [5 points] What is the after-tax real interest rate? Explain how consumption taxes affect it.

Answer

The after tax real interest rate is the relative price of consumption across periods. It can be read off the Euler equation as $\lambda'_k q' \frac{\lambda_c}{\lambda'_c}$. If consumption gets cheaper over time, the real interest rate rises.

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4. [10 points] Define a competitive equilibrium in sequence language. Assume that all government instruments except for one tax rate are given.

Answer

Objects: $\{c_t, l_t, k_t, w_t, q_t\}$ and one tax rate.

Equations:

- household: Euler, static, budget constraint. (3)
- firm: 2 standard FOCs.
- government: budget constraint.
- market clearing: goods (RC), labor (implicit), capital (implicit).

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5. [10 points] Find the constant tax rates that implement the first-best allocation. Explain the intuition. Does this work when g_t is time-varying?

Answer

The first-best allocation could be implemented with lump-sum taxes. Then all tax wedges disappear from the FOCs. This can be replicated by setting $\lambda_k = 1$ ($\tau_k = 0$) and $\lambda_c = \lambda_l$ or $\tau_c = -\tau_l$.

Not changing the intertemporal allocation requires zero capital tax rates and constant consumption taxes, so that the relative price of consumption does not change. Leaving the static allocation unchanged requires that the relative price of consumption vs leisure not be distorted. The government taxes labor, which makes leisure more attractive. This is offset by subsidizing consumption. The levels of τ_l and τ_c are determined by the revenue requirement g_t.

When g_t is time-varying, this fails.

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6. [5 points] Would it be possible to implement the equilibrium induced by any given (time varying) τ_l sequence using some sequence of (τ_c, τ_k)?

Answer

No. Just consider the case where λ_l is a constant. Replicating this with consumption taxes requires $\lambda_c = \lambda_l$. Not distorting the intertemporal allocation then requires $\tau_k = 0$. But that would not raise the correct amount of revenue. It truly takes all three taxes to implment a given allocation.

End of exam.