Macroeconomics Qualifying Examination

May 2019

Department of Economics

UNC Chapel Hill

Instructions:

- This examination consists of **4** questions. Answer all questions.
- The total number of points is 200.
- If you believe a question is ambiguously stated, ask for clarification. Unnecessary simplifications will be penalized.
- Do not consult any books, notes, calculators, or cell phones.
- **Write legibly**.
- Number your answers.
- Explain your answers.
- Budget your time wisely. Don't get hung up on one question.
- Good luck!

1 Vintage Capital

We study a version of the standard growth model where different capital vintages are used in production. A capital vintage is the capital produced at a particular date (rather like wine).

Demographics: There is a single infinitely lived household.

Preferences: $\sum_{t=0}^{\infty} \beta^t \ln c_t$ with $0 < \beta < 1$.

Endowments:

- At $t = 0$, the household owns X_s units of capital of each vintage $s < t = 0$.
- The household supplies one unit of labor in each period.

Technology:

- Total output is the sum of the outputs produced by all vintages: $Y_t = \sum_{s<t} Y_{s,t}$.
- The output produced by vintage s is given by $Y_{s,t} = \gamma^s \min\{K_{s,t}, L_{s,t}\}$ with $\gamma > 1$.
- $L_{s,t}$ is the amount of labor used with vintage s capital.
- Total labor supply is 1: $\sum_{s<t} L_{s,t} \leq 1$.
- $K_{s,t} \leq X_s$ is the amount of vintage s capital used in $t > s$.
- X_s is the amount of capital produced in period s. It never depreciates and can be used in production in all periods $t > s$.
- Output can be used for consumption and investment: $Y_t = c_t + \gamma^t X_t$. Note that the same γ^t affects the cost of investment and its productivity.

Markets: There are competitive markets for consumption (numeraire), labor rental (wage w_t), and capital rental ($q_{s,t}; s < t$).

Household: The household owns all capital and rents it to firms. The budget constraint is

$$c_t + \gamma^t X_t = \sum_{s<t} q_{s,t} K_{s,t} + w_t \tag{1}$$

because the equilibrium price of X_t must be γ^t. But we can simply think of the household as holding a complicated portfolio with rate of return R and budget constraint $a' = Ra - c$. Then we have the standard Euler equation $1/c = \beta R'/c'$.

Questions:

1. [10 points] Consider a firm that can only hire capital of a particular vintage $s < t$. Solve the firm's problem. Show that $q_{s,t} = \gamma^s - w_t$ for any vintage that operates in period t.

2. [6 points] Intuitively, what do you expect the age pattern of $K_{t-j,t}$ to look like(where j is the age of the vintage)? Explain your answer in words.

3. [5 points] Intuitively, what do you expect the age pattern of $q_{t-j,t}$ to look like (where j is the age of the vintage)? Explain your answer in words.

4. [7 points] Consider the marginal vintage that operates in t (the oldest one with $K_{s,t} > 0$). If it is T periods old, show that $\gamma^{t-T} = w_t$.

5. [4 points] The steady state interest rate is defined by $\gamma^s = \sum_{j=1}^{T} q_{s,s+j} R^{-j}$. Explain this in words. Note that this implies

$$1 = \sum_{j=1}^{T} R^{-j} \left[1 - \gamma^{j-T}\right] \tag{2}$$

   You do not have to derive (2).

6. [10 points] Find the growth rates of $q_{t-j,t}, w_t, c_t, Y_t, X_t, K_{t-j,t}$ on the balanced growth path, where j is a fixed vintage age. Assume that T is constant over time.

7. [7 points] Obtain 2 equations that implicitly solve for the balanced growth path values of R and T.

1.1 Answer:[1]

1. Firm:

   Take as given $q_{s,t}$ and w_t. The firm clearly sets $K_{s,t} = L_{s,t}$ and therefore solves $\max_{K_{s,t}} \gamma^s K_{s,t} - q_{s,t} K_{s,t} - w_t K_{s,t}$.

   Case 1: $\gamma^s = q_{s,t} + w_t$. The vintage can operate. The scale of the firm is indeterminate (as usual).

   Case 2: $\gamma^s < q_{s,t} + w_t$. The vintage cannot operate.

   Case 3: $\gamma^s > q_{s,t} + w_s$. The firm makes infinite profits. Cannot happen in equilibrium.

2. We expect $K_{s,t} = X_s$ for $s \geq t - T$ and $K_{s,t} = 0$ for older vintages. The constraint is labor supply. Because of the Leontief technology, each unit of capital needs one unit of labor. Only the most productive vintages can be used. Those are the newest.

[1] Based on UMN qualifying exam, spring 2007.

3. We expect $q_{t-j,t}/q_{t-j-1,t} = \gamma$ for all vintages that operate. This is driven by productivities. One unit of vintage j is equivalent to $1/\gamma$ units of $j-1$.

4. For the marginal vintage (the last one employed), $q_{t-T,t} = 0$. There is (generically) excess supply of capital. Apply the firm's optimality condition.

5. The cost of buying one unit of X_s equals the present value of revenues from renting it out. To derive $R(T)$: $\gamma^t = \sum_{j=1}^{T} R^{-j} \left[\gamma^t - \gamma^{j-T}\right]$. Divide through by γ^t.

6. Balanced growth rates: $q_{t-j,t} = \gamma^t \left[\gamma^{-j} - \gamma^{-T}\right]$ grows at rate γ. So does the wage. The resource constraint then requires that X_t and $K_{t-j,t}$ are constant over time. Consumption grows at γ, as does output.

7. Then we have R from the Euler equation $\beta R = 1 + g = \gamma$. Finally, $R(T)$ could be solved for T.

2 Lucas Asset Pricing

Demographics: A single representative agent who lives forever.

Preferences: $\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}u(c_t)$ with $0<\beta<1$ and $u(c)=c^{1-\sigma}/(1-\sigma)$.

Endowments:

- There is one tree which produces dividend stream d_t.
- The growth rate of dividends $g_t = d_t/d_{t-1}$ can take on the values $g_t=\gamma>1$ or $g_t=1$.
- The growth rate follows a Markov chain with transition probabilities $f(b,a)=\Pr(g'=b|g=a)$. Specifically:
  - $f(\gamma,\gamma)=\pi$ and thus $f(1,\gamma)=1-\pi$.
  - $f(1,1)=1$ and thus $f(\gamma,1)=0$. In words: $g=1$ is an absorbing state.

Markets: There are competitive markets for consumption (numeraire), trees (price $p(g,d)$), and Arrow securities for each state tomorrow (price $q(g',g)$; the prices turn out not to depend on d).

Questions:

1. [15 points] Write down the household's Bellman equation. Be sure to write the problem in a fully recursive way, suitable for defining a Recursive Competitive Equilibrium. Be sure to define the individual and aggregate state.
2. [8 points] Write down the Lucas asset pricing equations. These are standard, so you don't have to derive them. Substitute in the equilibrium value of the marginal rate of substitution.
3. [7 points] Derive the equilibrium prices of state contingent claims when $g=1$. Hint: $q(1,1)=\beta$.
4. [7 points] Do the same when $g=\gamma$. Hint: $q(\gamma,\gamma)=\pi\beta\gamma^{-\sigma}$.
5. [6 points] What would the interest rate on a risk-free bond be when $g=\gamma$?
6. [7 points] Derive and explain the ranking of expected returns of the two state contingent claims and the riskless bond when $g=\gamma$.

2.1 Answer: Lucas Asset Pricing

1. Bellman equation:

   The household's individual state is $s = (k, x(g))$ (where x is really a vector for all g). The aggregate state is $S = (g, d)$. The budget constraint is

$$p(S)\,k'(s,S) + \sum_{g'} q(g',g)\,x'(g',s,S) + c(s,S) = (p(S)+d)\,k + x(g) \tag{3}$$

   The Bellman equation is

$$V(s,S) = \max_{k'(s,S),c(s,S),x'(g,s,S)} u(c) + \beta \mathbb{E} V(s',S') \tag{4}$$

   subject to the budget constraint.

2. We always have

$$1 = \mathbb{E}\{MRS \times R'\} \tag{5}$$

   where the MRS is given by

$$MRS = \frac{\beta u'(c')}{u'(c)} = \beta (c/c')^{\sigma} \tag{6}$$

   In equilibrium, $c_t = d_t$ and therefore $c_{t+1}/c_t = g_{t+1}$. Hence,

$$1 = \mathbb{E}\left\{\beta (g')^{-\sigma} R'\right\} \tag{7}$$

   For the tree, this is standard with $R' = (p' + d')/p$. For the state contingent claim:

$$q(g',g) = \Pr(g'|g)\,\beta (g')^{-\sigma} \tag{8}$$

$$= f(g',g)\,\beta (g')^{-\sigma} \tag{9}$$

3. When $g = 1$ we know for sure that $g' = 1$. Then $q(\gamma, 1) = 0$ and $q(1,1) = \beta$. We now live in a risk-free world where a state-contingent claim is the same as a discount bond. The interest rate is β^{-1}.

4. When $g = \gamma$, we have

   (a) $q(\gamma,\gamma) = \pi\beta\gamma^{-\sigma}$

   (b) $q(1,\gamma) = (1-\pi)\,\beta$.

5. A risk-free bond costs the same as the two state-contingent claims. Therefore,

$$R^{-1} = \pi\beta\gamma^{-\sigma} + (1-\pi)\,\beta \tag{10}$$

   Or use the standard condition $1 = \beta\mathbb{E}\left\{(g')^{-\sigma}\right\} R'$.

6. The expected return on the state contingent claims is $f(g',g)/q(g',g)$. For $g' = \gamma$: $(\beta\gamma^{-\sigma})^{-1} > \beta^{-1}$. For $g' = 1$: β^{-1}. For the bond: in between those two. Intuition: payouts are most valuable when consumption growth is small ($g' = 1$). Expected returns relate inversely to payouts.

3 Analytical Solution of an RBC Model

Setup

- Demographics: There is a representative infinitely lived household.
- Endowments: The household is endowed with an initial capital stock K_0 and supplies labor each period.
- Preferences: Lifetime utility is given by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - \chi N_t] \tag{11}$$

  where $\beta \in (0,1)$ is a discount factor, C_t denotes consumption, N_t denotes hours worked, and χ captures the disutility from working.

- Technology: The aggregate resource constraint is

$$Y_t = I_t + C_t \tag{12}$$

  where

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{13}$$

  and A_t is productivity. We assume full depreciation so that capital evolves according to

$$K_{t+1} = I_t. \tag{14}$$

- Markets: All markets are competitive. The final good is the numeraire. The real rental rate of capital is R_t. The real wage is W_t. There is no bond. Firms rent capital and labor from the household each period. All profits of the firm are distributed to the household who owns the firm.

Questions

1. [10 points] State the problem of the representative household and derive the FOCs.
2. [4 points] State the problem of the representative firm and derive the FOCs.
3. [5 points] Derive the Euler equation

$$\frac{1}{C_t} = \beta E_t[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{1}{C_{t+1}}] \tag{15}$$

4. [13 points] Guess that in equilibrium investment is a constant fraction of output, i.e. $I_t = \gamma Y_t$. and $C_t = (1-\gamma)Y_t$. Verify that the guess is correct and find γ.

   Hint: Start with the Euler equation, then use the guess.

5. [4 points] Show that in equilibrium $N = \frac{(1-\alpha)}{(1-\beta\alpha)\chi}$.

6. [8 points] Show that the equilibrium process of output can be written as

$$y_t = a_t + \alpha y_{t-1} \tag{16}$$

   where $y_t \equiv \log Y_t$, $a_t \equiv \log A_t$, and we dropped constants.

7. [5 points] Assume the economy is in the steady state with $a = 0$. Now there is a shock to productivity in period 0 that is unanticipated and transitory:

$$a_t = \begin{cases} 1 & \text{if } \ t = 0 \\ 0 & \text{if } \ t \neq 0 \end{cases} \tag{17}$$

   Compute the response in log output for the periods $t = 0, 1, 2, 3$.

3.1 Solution

1. The Lagrangian of the representative household is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - \chi N_t + \lambda_t (C_t + K_{t+1} - W_t N_t - R_t K_t)] \tag{18}$$

   The FOC are

$$C_t : \frac{1}{C_t} = \lambda_t \tag{19}$$

$$N_t : \chi = \lambda_t W_t \tag{20}$$

$$K_{t+1} : \lambda_t = \beta E_t [R_{t+1} \lambda_{t+1}] \tag{21}$$

   and a transversality condition

$$\lim_{j \to \infty} \beta^j E_t \lambda_{t+j} K_{t+j+1} = 0 \tag{22}$$

2. The firm problem is static

$$\max \Pi_t = A_t K_t^\alpha N_t^{1-\alpha} - W_t N_t - R_t K_t \tag{23}$$

   The FOC are

$$R_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} = \alpha \frac{Y_t}{K_t} \tag{24}$$

$$W_t = (1-\alpha) A_t K_t^\alpha N_t^{-\alpha} = (1-\alpha) \frac{Y_t}{N_t} \tag{25}$$

3. Equations (19), (21), and (24) yield

$$\frac{1}{C_t} = \beta E_t[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{1}{C_{t+1}}]. \tag{26}$$

4. Constant saving rate:

$$\frac{1}{C_t} = \beta E_t \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{1}{C_{t+1}} \right] \tag{27}$$

$$\frac{1}{(1-\gamma)Y_t} = \beta E_t \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{1}{(1-\gamma)Y_{t+1}} \right] \tag{28}$$

$$= \beta E_t \left[\alpha \frac{1}{(1-\gamma)K_{t+1}} \right] \tag{29}$$

$$K_{t+1} = I_t = \alpha\beta Y_t \tag{30}$$

Hence, investment is a constant fraction of output with $\gamma = \beta\alpha$, which verifies the guess.

5. The first order conditions imply:

$$\frac{1}{C_t} = \frac{\chi}{W_t} \tag{31}$$

$$\frac{1}{(1-\beta\alpha)Y_t} = \frac{\chi}{(1-\alpha)\frac{Y_t}{N_t}} \tag{32}$$

$$N = \frac{(1-\alpha)}{(1-\beta\alpha)\chi} \tag{33}$$

6. Use $N_t = \frac{(1-\alpha)}{(1-\beta\alpha)\chi}$ and $K_t = I_{t-1} = \beta\alpha Y_{t-1}$ in the production function to obtain

$$Y_t = A_t(\beta\alpha Y_{t-1})^\alpha \left(\frac{(1-\alpha)}{(1-\beta\alpha)\chi} \right)^{1-\alpha} \tag{34}$$

Taking logs and dropping constants

$$y_t = a_t + \alpha y_{t-1} \tag{35}$$

7. Labor is constant, $N = \frac{(1-\alpha)}{(1-\beta\alpha)\chi}$. The FOC of the firm yields

$$W = (1-\alpha)\frac{Y_t}{N_t} \tag{36}$$

$$w_t = y_t = a_t + \alpha y_{t-1} \tag{37}$$

Employment is constant.

Output and wages will follow

$$\{y_t\}_{t=0}^{\infty} = \{w_t\}_{t=0}^{\infty} = \{1, \alpha, \alpha^2, \alpha^3, \dots\} \tag{38}$$

4 Optimal Monetary Policy in a New Keynesian Model

Setup We consider a standard New Keynesian model with sticky prices. The economy is described by the following log-linearized system.

$$\tilde{x}_t = E_t\tilde{x}_{t+1} - (\tilde{i}_t - E_t\tilde{\pi}_{t+1}) \tag{39}$$
$$\tilde{\pi}_t = \beta E_t\tilde{\pi}_{t+1} + \gamma\tilde{x}_t + e_t \tag{40}$$

where

- $\tilde{x}_t = \frac{x_t - x}{x}$ is the output gap in (relative) deviations from its steady state value x,
- $\tilde{i}_t = i_t - i$ is the nominal interest rate in deviations from its steady state value i,
- $\tilde{\pi}_t = \pi_t - \pi$ is inflation in deviations from its (zero-inflation) steady state value $\pi = 0$. Inflation is defined as $\pi_t = \frac{P_t}{P_{t-1}} - 1$, where P_t is the price level.
- $\gamma > 0$ is a constant.
- e_t is a cost shock and we assume that

  $$e_t = \rho_e e_{t-1} + \varepsilon_t, \tag{41}$$

  where ε_t is a white noise process and $\rho_e \in (0,1)$.

The objective function of the central bank is

$$\min \frac{1}{2}E_0\left(\sum_{t=0}^{\infty}\beta^t\left(\tilde{\pi}_t^2 + \omega\tilde{x}_t^2\right)\right) \tag{42}$$

where ω denotes the weight that the central bank puts on the output gap (relative to inflation). We take the expected inflation at time zero, $E_{-1}\tilde{\pi}_0$, as given.

We assume that the central bank sets policy with commitment. This amounts here to choosing state-contingent policies $\{\tilde{\pi}_t, \tilde{x}_t\}_{t=0}^{\infty}$ to maximize equation (42) subject to the New Keynesian Phillips Curve (40).

Questions

1. [12 points] Set up the Lagrangian of the central bank problem and derive the first order conditions with respect to $\tilde{\pi}_0, \tilde{\pi}_t, \tilde{x}_t$ (recall that $E_{-1}\tilde{\pi}_0$ is taken as given in period 0).

2. [18 points] Show that the central bank's optimal policy implies

$$E_0\tilde{x}_t = -\frac{\gamma}{\omega}\sum_{j=0}^{t} E_0\tilde{\pi}_{t-j}. \tag{43}$$

Hint: You can first use the FOCs to show that $E_t\tilde{x}_{t+1} = \tilde{x}_t - \frac{\gamma}{\omega}E_t\tilde{\pi}_{t+1}$ $\forall t$. Then, start in the initial period and iterate forward.

3. [17 points] Use the previous result to show that the central bank's optimal policy implies

$$\tilde{x}_t = -\frac{\gamma}{\omega}\tilde{P}_t, \tag{44}$$

where $\tilde{P}_t = \frac{P_t - P}{P}$, and P is the steady state price level.

Hint: Use the approximation $\tilde{\pi}_t \approx \log(1+\tilde{\pi}_t)$ and the definition of the inflation rate, $\pi_t = \frac{P_t}{P_{t-1}} - 1$. Furthermore, you can use $\tilde{P}_t = \frac{P_t - P}{P} \approx \log(P_t) - \log(P)$. The economy is initially in the zero-inflation steady state, hence there is no initial price deviation, i.e. $\tilde{P}_{-1} = 0$.

4. [5 points] Is there a trade-off between inflation stabilization and zero output gap in this version of the model? Explain your answer.

4.1 Solution

1. The Lagrangian is

$$\mathcal{L} = E_0\sum_{t=0}^{\infty}\beta^t\Bigg(-\frac{1}{2}\left(\tilde{\pi}_t^2 + \omega\tilde{x}_t^2\right) \tag{45}$$

$$+ \lambda_t\left(\tilde{\pi}_t - \gamma\tilde{x}_t - \beta E_t\tilde{\pi}_{t+1} - e_t\right)\Bigg) \tag{46}$$

We take as given the expected inflation at time zero $E_{-1}\tilde{\pi}_0$.

The first order conditions are

$$\frac{\partial\mathcal{L}}{\partial\tilde{\pi}_0} = 0 \Leftrightarrow \tilde{\pi}_0 = \lambda_0 \tag{47}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{\pi}_t} = 0 \Leftrightarrow -\beta^{t-1}\lambda_{t-1}\beta - E_{t-1}\beta^t\tilde{\pi}_t + E_{t-1}\beta^t\lambda_t = 0 \tag{48}$$

$$\Rightarrow E_{t-1}\tilde{\pi}_t = E_{t-1}\lambda_t - \lambda_{t-1}\ \forall t > 0 \tag{49}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{x}_t} = 0 \Leftrightarrow \omega\tilde{x}_t = -\lambda_t\gamma \Leftrightarrow \tilde{x}_t = -\lambda_t\frac{\gamma}{\omega} \Leftrightarrow -\tilde{x}_t\frac{\omega}{\gamma} = \lambda_t\ \forall t \tag{50}$$

2. Combining the first and last FOC yields

$$\tilde{x}_0 = -\frac{\gamma}{\omega}\tilde{\pi}_0 \tag{51}$$

The second FOC yields

$$E_{t-1}\tilde{\pi}_t = E_{t-1}\lambda_t - \lambda_{t-1} \tag{52}$$

$$E_t\tilde{x}_{t+1} = \tilde{x}_t - \frac{\gamma}{\omega}E_t\tilde{\pi}_{t+1}\ \forall t \tag{53}$$

We start in the initial period and iterate forward.

$$\begin{aligned}
E_0\tilde{x}_1 &= \tilde{x}_0 - \frac{\gamma}{\omega}E_0\tilde{\pi}_1 && (54)\\
&= -\frac{\gamma}{\omega}\tilde{\pi}_0 - \frac{\gamma}{\omega}E_0\tilde{\pi}_1 && (55)\\
E_0\tilde{x}_2 &= \tilde{x}_1 - \frac{\gamma}{\omega}E_0\tilde{\pi}_2 && (56)\\
&= -\frac{\gamma}{\omega}\tilde{\pi}_0 - \frac{\gamma}{\omega}E_0\tilde{\pi}_1 - \frac{\gamma}{\omega}E_0\tilde{\pi}_2 && (57)
\end{aligned}$$

Continuing in this fashion yields for period t

$$E_0\tilde{x}_t = -\frac{\gamma}{\omega}\sum_{j=0}^{t}E_0\tilde{\pi}_{t-j} \tag{58}$$

3. From the definition of the inflation rate we get

$$\begin{aligned}
1+\pi_t &= \frac{P_t}{P_{t-1}} && (59)\\
\pi_t &\approx \log(1+\pi_t) = \log P_t - \log P_{t-1} && (60)
\end{aligned}$$

Since $\tilde{\pi}_t = \pi_t - \pi$ and $\pi = 0$, we get

$$\tilde{\pi}_t \approx \log P_t - \log P_{t-1} \tag{61}$$

The sum therefore becomes

$$E_0\tilde{x}_t = -\frac{\gamma}{\omega}\sum_{j=0}^{t}E_0\tilde{\pi}_{t-j} = -\frac{\gamma}{\omega}E_0(\log P_t - \log P_{-1}) \tag{62}$$

Using $\tilde{P}_t = \frac{P_t - P}{P} \approx \log(P_t) - \log(P)$, we get

$$E_0\tilde{x}_t = -\frac{\gamma}{\omega}E_0(\tilde{P}_t - \tilde{P}_{-1}) \tag{63}$$

The initial price deviation is zero, hence we can write this as

$$\tilde{x}_t = -\frac{\gamma}{\omega}\tilde{P}_t. \tag{64}$$

4. Yes, there is a trade-off. Because of the cost shock e_t , the central bank cannot at the same time achieve zero inflation and zero output gap in every period. Hence, there is a trade-off between inflation and output gap.

End of exam.