Macroeconomics Qualifying Examination

August 2011

Department of Economics

UNC Chapel Hill

Instructions:

- This examination consists of **three** questions. Answer all questions.
- Answering only two questions (or providing a cursory answer to the third question) will greatly reduce your chances of passing the exam.
- If you believe a question is ambiguously stated, ask for clarification. Unnecessary simplifications will be penalized.
- Do not consult any books, notes, calculators, or cell phones.
- **Write legibly**.
- Good luck!

1 Two Trees in a Lucas Model

Consider a standard Lucas Fruit Tree model with 2 types of trees.

- Demographics: There is a representative consumer who lives forever.
- Preferences:
$$E \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{1}$$
- Endowments: The household is endowed with one tree of each type i; $i \in \{1, 2\}$. A type i tree yields d_{it} units of the consumption good in period t. d_{it} is an i.i.d. random variable. Dividends take on the values d_L with probability π_i and $d_H > d_L$ with probability $(1 - \pi_i)$.
- Market arrangements: Households trade in competitive markets: (a) goods at price 1, (b) trees at prices p_{it}. p_{it} is "ex dividend" - it values the type i tree that has already paid its period t dividend.

Questions:

1. State the household's dynamic program. Be careful about the state variables. This is a model with aggregate uncertainty.
2. Derive the first-order conditions and Euler equations.
3. Define a Recursive Competitive Equilibrium.
4. Solve for the price of a type-1 tree as a function of the aggregate state. We are looking for a closed form solution that depends only on primitives.
5. Explain how the price of a type-1 tree depends on the current values of d_{it} for $i = 1, 2$. Provide the economic intuition.
6. Derive an expression for the price of an Arrow security that pays in a particular state $S_{t+1} = (d_{1,t+1}, d_{2,t+1})$. Apply this formula to derive the price for the Arrow security that pays in state $S_{t+1} = (d_L, d_H)$ when the current state is $S_t = (d_L, d_L)$.

2 Optimal Taxation with Two Consumption Goods

Demographics: There is a single representative consumer who lives forever.

Endowments: The consumer is endowed with k_0 units of the consumption / capital good at time 0.

Preferences: The consumer values consumption of 2 goods according to

$$\max \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(g_t)\}$$

Technology: A single good is used for consumption, government consumption G_t, and capital accumulation K_t. The resource constraint is given by

$$F(K_t, L_t) + (1 - \delta) K_t = c_t + \varphi\, g_t + G_t + K_{t+1} \tag{2}$$

where $\varphi > 0$ denotes the rate at which the good can be converted into the g consumption good. F has constant returns to scale.

Government: The government imposes consumption taxes at rates τ_{ct} and τ_{gt}, respectively. Tax revenues are used to purchase G_t.

Market arrangements: There is a single representative firm. The household rents one unit of labor to the firm at wage w_t. He also rents capital to the firm at rental rate q_t, earning the gross return $R_t = q_t + 1 - \delta$. He buys consumption goods from the firm. All markets are competitive.

Questions:

1. Set up a Dynamic Program for the household problem. Define a solution. The budget constraint is given by

$$k_{t+1} = w_t + R_t k_t - c_t (1 + \tau_{ct}) - g_t\, p_t (1 + \tau_{gt})$$

   where p is the price of g.

2. Define a competitive equilibrium. Assume that the tax rate τ_{ct} is set to balance the government budget.

3. Derive 4 equations that implicitly solve for the steady state values of c, g, k and R.

4. Derive an expression for the ratio of tax rates, $(1 + \tau_g)/(1 + \tau_c)$, that maximizes steady state welfare. *Warning*: This question contains an unintended trap. *Hint:* Think before taking long derivatives!

3 Monetary Policy

This question is based on Barro and Gordon's analysis of policymakers having discretion versus commitment in policymaking. We will analyze two such objective functions.

Time is discrete and goes on forever. Time subscripts are omitted below for simplicity. The aggregate supply function and inflation are governed by:

$$y = y_n + a(\pi - \pi^e) + e$$

$$\pi = \Delta m + v$$

where y is the output level, y_n is the natural level of output, π denotes the level of inflation and π^e denotes the expected value of inflation. e and v are i.i.d supply and velocity shocks respectively. Note also that $E(e) = 0$, $E(e)^2 = \sigma_e^2$, $E(v) = 0$ and $E(v)^2 = \sigma_v^2$. The central bank sets money supply growth, Δm, as its instrument.

The central bank objective function is given by

$$U = \lambda(y - y_n) - (1/2)\pi^2$$

where $\lambda > 0$ is a parameter. The timing of the model is as follows:

1. In the case of commitment: The central bank chooses a rule that governs Δm.
2. Households makes decisions: they form inflation expectations, π^e.
3. The supply shock e is realized.
4. The monetary authority chooses Δm. In the case of commitment: Δm follows the policy rule.
5. The velocity shock v is realized
6. y, π are determined.

Questions Part A:

1. Explain the intuition or conditions under which an aggregate supply curve like the one above could arise.
2. Assume that the central bank cannot commit.
   (a) With this policymaker's utility function is there an inflation bias? Demonstrate.
   (b) Derive an expression for the expected value of Central Bank's utility function.
3. Now, suppose the central bank commits to zero money supply growth rule, i.e.: $\Delta m = 0$. Derive an expression for the expected value of Central Bank's utility function.
4. Compare the discretionary and commitment outcomes.

Questions Part B: Now suppose that policy objective is described by the **loss** function:

$$V = (1/2)\lambda(y - y_n - k)^2 + (1/2)\pi^2$$

1. Compare this loss function to the linear-in-output utility function above. You should demonstrate whether one specification encompasses the other and also highlight any key differences between the two functions.

2. Give reasons why the parameter k would appear in the loss function?

3. Assume that the central bank cannot commit.

   (a) Demostrate whether there is inflation bias under this loss function.

   (b) Derive an expression for the expected value of the loss function.

4. Suppose the central bank commits to, $\Delta m = b_0 + b_1 e$. Find the values of b_0 and b_1 minimize the unconditional expectation of the loss function? Compare the expected loss to the discretionary case. **Note:** *Under commitment policy, the central bank commits itself to particular values of the parameters b_0 and b_1 prior to the formation of expectations by the public and prior to observing the particular realization of the shock e.*

4 Answers

4.1 Answer: Two Lucas Trees[1]

1. The aggregate state is $S_t = (d_{1t}, d_{2t})$. The individual state is $s_t = (k_{1,t}, k_{2t})$. The budget constraint is given by

$$\sum_i k_{it}(p_{it} + d_{it}) = \sum_i k_{it+1}p_{it} + c_t$$

The Bellman equation is

$$V(k_1, k_2; S) = \max \log \left(\sum_i k_i(p_i + d_i) - k_i' p_i \right) + \beta \mathbb{E} V(k_1', k_2', S')$$

2. First-order conditions: We get the usual Lucas asset pricing equations

$$p_i/c = \beta \mathbb{E} \frac{p_i' + d_i'}{c'} \tag{3}$$

3. Recursive CE: Objects: value function and policy functions $k_i' = f_i(s, S)$ and value function and price functions $p_i(S)$ that satisfy:

- household: V is a fixed point and f_i solve the max.
- asset market clearing: $k_i' = 1$.
- goods market clearing: $c = \sum_i d_i$.
- no need for the usual consistency condition: the aggregate state is exogenous.

4. Price of the tree: Forward iteration on the asset pricing condition implies the usual pricing equation

$$p_{i,t} = (\sum_j d_{jt}) \mathbb{E} \sum_{s=1}^{\infty} \beta^s \frac{d_{i,t+s}}{\sum_j d_{j,t+s}} = (\sum_j d_{jt}) \frac{\beta}{1-\beta} X_i$$

where X_i is a constant that depends on parameters.

5. Intuition: The entire term in the expectation is the same for all t. Therefore, a unit increase in the dividend of either tree raises the price of the type i tree by the same amount. This is simply a consequence of iid dividends. Higher d_{it} implies lower marginal utility today, but does not change anything in the future. Agents want to save more and drive up the value of the tree.

[1] Based on a question due to Tony Smith.

6. Arrow securities: The general pricing formula is

$$\frac{q(d_1', d_2')}{d_1 + d_2} = \beta \frac{\Pr(d_1') \Pr(d_2')}{d_1' + d_2'} \tag{4}$$

where $\Pr(d_i')$ is π_i or $1 - \pi_i$. For the requested case:

$$\frac{q(d_L, d_H)}{2d_L} = \beta \frac{\pi_1 (1 - \pi_2)}{d_L + d_H} \tag{5}$$

4.2 Answer: Optimal taxation with two consumption goods

1. The Bellman equation is

$$V(k) = \max\ u(c) + v(g) + \beta\, V(k') + \lambda\ \{w + R\, k - c\,(1 + \tau_c) - g\, p\,(1 + \tau_g) - k'\}$$

First-order conditions are

$$\begin{aligned} u'(c) &= \lambda\,(1 + \tau_c) \\ v'(c) &= \lambda\, p\,(1 + \tau_g) \\ \beta\, V'(k') &= \lambda \end{aligned}$$

After a bit of manipulation, we arrive at the following solution: A set of sequences (c_t, g_t, k_t) that solve

$$\frac{v'(g)}{u'(c)} = p\, \frac{1 + \tau_g}{1 + \tau_c} \tag{6}$$

$$u'(c) = \beta\, R'\, u'(c')\, \frac{1 + \tau_c}{1 + \tau_c'}$$

together with the budget constraint and two boundary conditions: k_0 given and the TVC $\lim_{t \to \infty} \beta^t\, u'(c_t)\, k_t = 0$.

2. A competitive equilibrium is an allocation $(c_t, g_t, k_t, \tau_{ct}, K_t, L_t)$ and a price system (r_t, w_t, p_t, R_t) that satisfy:

- 3 household conditions
- 2 firm first-order conditions and $p_t = \varphi$.
- The government budget constrain $G_t = \tau_{ct}\, c_t + \tau_{gt}\, p_t\, g_t$
- Market clearing: $K_t = k_t$; $L_t = 1$; and goods market clearing (2).
- The identity $R_t = 1 - \delta + r_t$.

3. The Euler equation fixes the interest rate at $R_{ss} = 1/\beta$. The capital stock is then determined by $R_{ss} = 1 - \delta + f'(k_{ss})$. The static first-order condition (6) together with goods market clearing,

$$y \equiv f(k_{ss}) - \delta\, k_{ss} - G = c_{ss} + \varphi\, g_{ss} \tag{7}$$

then determines c_{ss} and g_{ss}.

4. Since the capital stock is not affected by taxation, the government maximizes $u(c) + v(g)$ subject to the static first-order condition (6), goods market clearing (7), and the government budget constraint.

The Lagrangean is given by

$$\max_{g,\tau_g} u(y - \varphi\, g) + v(g) + \lambda \left\{ v'(g) \left[1 + \frac{G - \tau_g\, \varphi\, g}{y - \varphi\, g} \right] - \varphi\, (1 + \tau_g)\, u'(y - \varphi\, g) \right\} \tag{8}$$

The c's have been substituted out using $c = y - \varphi g$.

The constraint in the braces is the static FOC where the government budget constraint has been used to replace τ_c by $[G - \tau_g\, \varphi\, g]/c$.

Taking the first-order condition with respect to τ_g leads to the unexpected trap:

$$\lambda \left\{ v'(g) \frac{\varphi\, g}{y - \varphi\, g} + \varphi\, u'(y - \varphi\, g) \right\} = 0 \tag{9}$$

The term in the $\{\}$ must be strictly positive. But then λ must be 0!

Lagrange's Theorem: "If $(g,\, \tau_g)$ maximize (8), then there exists a λ such that (9) holds (amongs other conditions)."

This λ happens to be zero here. This neither implies that the constraint is not binding (though it *usually* has that interpretation), nor that any τ_g maximizes (8).

Why does this happen here? Tax rates have no effect other than to change the ratio c/g. *Any* c/g is attainable. Hence, the first-best allocation is attainable, which is the allocation that maximizes the *unconstrained* version of (8).

This is also the result we obtain when we proceed mechanically. Taking the first-order condition for g and imposing $\lambda = 0$ yields

$$\varphi\, u'(c) = v'(g)$$

The tax rates that implement this can be backed out from the static condition (6): $\tau_g = \tau_c$. The fact that the tax rates are the same, regardless of the utility function, appears surprising at first. A fundamental principle of optimal taxation indicates to tax goods with lower demand elasticities more heavily. But this does not apply here because the two consumption taxes together are equivalent to a lump-sum tax and therefore first-best.

4.3 Answer: Monetary Policy

Part A

1. Motivate the aggregate supply from say one-period nominal wage contracts set at the start of each period based on the public expectation of the rate of inflation. If $\pi > \pi^e$ real wage is reduced and firms expand employment, leading to increased output.

2. Plug in

$$\pi = \Delta m + v \tag{10}$$
$$y = y_n + a(\pi - \pi^e) + e \tag{11}$$

into the utility function

$$U = \lambda(y - y_n) - \frac{1}{2}\pi^2 \tag{12}$$
$$= \lambda[a(\Delta m + v - \pi^e) + e] - \frac{1}{2}(\Delta m + v)^2 \tag{13}$$

Taking first order conditions with respect to Δm we get

$$a\lambda - \Delta m = 0 \tag{14}$$
$$\Delta m = a\lambda > 0 \tag{15}$$

Actual inflation equals $a\lambda + v$. Private households understand the CB's incentives thus

$$\pi^e = E[\Delta m] = a\lambda \tag{16}$$

average inflation fully anticipated (therefore no effect on output) - economy has positive inflation with no benefit. So yes there is an inflation bias. The expected utility under this discretionary policy is

$$E(U) = E\left[\lambda(av + e) - \frac{1}{2}(a\lambda + v)^2\right] \tag{17}$$

Terms linear in the shocks fall out giving

$$E(U) = -\frac{1}{2}\left[a^2\lambda^2 + \sigma_v^2\right] \tag{18}$$

where σ_v^2 is the variance of the velocity shock.

3. Commiting to $\Delta m = 0$ theis implies that $\pi^e = E[\Delta m] = 0$ this yields

$$U = \lambda[av + e] - \frac{1}{2}v^2 \tag{19}$$

therefore $E(U) = -\frac{1}{2}\sigma_v^2$

4. Compare to the discretionary case $-\frac{1}{2}\sigma_v^2 > -\frac{1}{2}[a^2\lambda^2 + \sigma_v^2]$ therefore commiting to zero money growth is better from a welfare perspective.

Part B

1. Now lets use

$$\begin{aligned} V &= \frac{1}{2}\lambda\left(y - y_n - k\right)^2 + \frac{1}{2}\pi^2 && (20) \\ &= -\lambda k\left(y - y_n\right) + \frac{1}{2}\pi^2 + \frac{1}{2}\lambda\left(y - y_n\right)^2 + \frac{1}{2}\lambda k^2 && (21) \end{aligned}$$

Notice that the signs on the $(y - y_n)$ and π^2 are reversed compared to the utility case - one is utility the other a loss function. There is an extra $\frac{1}{2}\lambda k^2$ which is just a constant so won't change the optimization. However, there is a output volatility component $\frac{1}{2}\lambda\left(y - y_n\right)^2$ which makes stabilizing the (volatility of) output shock e an option.

2. Plug in

$$\begin{aligned} \pi &= \Delta m + v && (22) \\ y &= y_n + a\left(\pi - \pi^e\right) + e && (23) \end{aligned}$$

into loss function

$$V = \frac{1}{2}\lambda\left[a\left(\Delta m + v - \pi^e\right) + e - k\right]^2 + \frac{1}{2}\left(\Delta m + v\right)^2 \quad (24)$$

taking F.O.C. with respect to money growth (conditional on e and π^e, and noting that v unknown at time of optimization)

$$a\lambda\left[a\left(\Delta m - \pi^e\right) + e - k\right] + \Delta m = 0 \quad (25)$$

we need to solve find $\pi^e = E(\Delta m)$. First solve for Δm

$$\Delta m = \frac{a^2\lambda\pi^e + a\lambda\left(k - e\right)}{1 + a^2\lambda} \quad (26)$$

taking expectation

$$\pi^e = E(\Delta m) = \frac{a^2\lambda\pi^e + a\lambda k}{1 + a^2\lambda} \quad (27)$$

Solve for π^e

$$\pi^e = a\lambda k > 0 \quad (28)$$

Inflation bis exist (notice it if just a multiple k away from the linear utility case.

3.

$$V = \frac{1}{2}\lambda\left[a\left(\frac{a^2\lambda\pi^e + a\lambda\left(k - e\right)}{1 + a^2\lambda} + v - \frac{a^2\lambda\pi^e + a\lambda k}{1 + a^2\lambda}\right) + e - k\right]^2 + \frac{1}{2}\left(\frac{a^2\lambda\pi^e + a\lambda\left(k - e\right)}{1 + a^2\lambda} + v\right)^2 \quad (29)$$

where $\pi^e = a\lambda k$

$$V = \frac{1}{2}\lambda\left[\frac{-a^2\lambda e}{1+a^2\lambda} + e + av - k\right]^2 + \frac{1}{2}\left[\frac{a^2\lambda(a\lambda k) + a\lambda(k-e)}{1+a^2\lambda} + v\right]^2 \tag{30}$$

$$= \frac{1}{2}\lambda\left[\frac{e}{1+a^2\lambda} + av - k\right]^2 + \frac{1}{2}\left[a\lambda k - \frac{a\lambda e}{1+a^2\lambda} + v\right]^2 \tag{31}$$

taking expectations

$$E(V^D) = \frac{1}{2}\lambda\left(\frac{1}{1+a^2\lambda}\right)^2\sigma_e^2 + \frac{1}{2}\lambda a^2\sigma_v^2 + \frac{1}{2}\lambda k^2 + \frac{1}{2}a^2\lambda^2k^2 + \frac{1}{2}\left(\frac{a\lambda}{1+a^2\lambda}\right)^2\sigma_e^2 + \frac{1}{2}\sigma_v^2 \tag{32}$$

$$= \frac{1}{2}\lambda\left(1+a^2\lambda\right)k^2 + \frac{1}{2}\frac{\lambda}{(1+a^2\lambda)}\sigma_e^2 + \frac{1}{2}\left(1+a^2\lambda\right)\sigma_v^2 \tag{33}$$

4. Commit to $\Delta m = b_0 + b_1 e$. Using FOC

$$a\lambda\left[a\left(\Delta m - \pi^e\right) + e - k\right] + \Delta m = 0 \tag{34}$$

and the fact that $\pi^e = E(\Delta m) = b_0$ we get

$$a\lambda\left[a\left(b_0 + b_1 e - b_0\right) + e - k\right] + b_0 + b_1 e = 0 \tag{35}$$

$$a^2\lambda b_1 e + b_1 e + b_0 = a\lambda k - a\lambda e \tag{36}$$

equating coefficients on e

$$b_1\left(1+a^2\lambda\right) = -a\lambda \tag{37}$$

$$b_1 = \frac{-a\lambda}{(1+a^2\lambda)} \tag{38}$$

optimal to set $b_0 = 0$ this implies that

$$\Delta m = b_0 + b_1 e = \frac{-a\lambda}{(1+a^2\lambda)}e \tag{39}$$

Compare to discretion

$$V = \frac{1}{2}\lambda\left[a\left(b_1 e + v\right) + e - k\right]^2 + \frac{1}{2}\left(b_1 e + v\right)^2 \tag{40}$$

$$= \frac{1}{2}\lambda\left[\frac{-a^2\lambda}{(1+a^2\lambda)}e + e + av - k\right]^2 + \frac{1}{2}\left[\frac{-a\lambda}{(1+a^2\lambda)}e + v\right]^2 \tag{41}$$

$$E(V^C) = \frac{1}{2}\frac{\lambda}{(1+a^2\lambda)}\sigma_e^2 + \frac{1}{2}\left(1+a^2\lambda\right)\sigma_v^2 + \frac{1}{2}\lambda k^2 \tag{42}$$

Compare to the discretionary case

$$E(V^D) - E(V^C) = \frac{1}{2}a^2\lambda^2k^2 > 0 \tag{43}$$

End of exam.