Midterm Exam. Econ720. Spring 2009

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 1:30 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

---

1 OLG Model with Human Capital

Consider the following two period OLG model.

Demographics: Households live for two periods. $N_t = (1+n)^t$ young households are born at date t.

Endowments: Each young is endowed with human capital h_t and assets b_t (in units of the good). b_t is an inheritance from the parents.

Preferences: Parents enjoy their own consumption and their children's utility. Preferences are recursively defined by

$$V(b_t, h_t) = u\left(c_t^y, c_{t+1}^o\right) + \beta\ V(b_{t+1}, h_{t+1}) \tag{1}$$

Technologies: Firms rent capital and labor from households at rental prices q_t and w_t, respectively. Firms produce with a constant returns to scale production function, $Y_t = F(K_t, L_t)$, and maximize period profits. Capital depreciates at rate δ.

A young household divides his time between work (l_t) and education ($1 - l_t$). Old households have human capital $h_{t+1}^o = g(1 - l_t, h_t)$.

New agents "inherit" human capital from their parents:

$$h_{t+1} = \varphi h_{t+1}^o \tag{2}$$

and parents understand that their children benefit from their human capital.

Questions:

(a) [10 points] State the **household**'s Bellman equation. The young budget constraint is

$$l_t h_t w_t + b_t = c_t^y + k_{t+1} \tag{3}$$

The old budget constraint is

$$R_{t+1} k_{t+1} + h_{t+1}^o w_{t+1} = c_{t+1}^o + b_{t+1}(1+n) \tag{4}$$

where b_{t+1} is a bequest and k_{t+1} denotes saving. R is the interest rate. w is the wage rate.

(b) [15 points] Derive the first order conditions and envelope conditions.

(c) [15 points] Interpret the first-order conditions. State the feasible perturbations that underly each first-order condition.

(d) [10 points] State the **market clearing** conditions.

(e) [15 points] Derive 3 equations that solve for the steady state values of R, h, l.

2 Habit formation

Consider the following decision problem. Preferences: $\int_0^\infty e^{-\theta t} U(c_t, h_t)\, dt$. Laws of motion: $\dot{k}_t = Ak_t - c_t$ and $\dot{h}_t = \rho(c_t - h_t)$.

(a) [5 points] Write down the current value Hamiltonian.

(b) [12 points] Derive the first-order conditions. Do not try to substitute out the costates.

(c) [8 points] Define a solution to this problem.

(d) [10 points] Derive the balanced growth rates of c, h, k. Assume that $U(c, h) = \frac{[c/h]^{1-\sigma}}{1-\sigma}$.

3 Answers

3.1 Answer: OLG model with human capital

(a) The Bellman equation is

$$\begin{aligned} V(b,h) &= \max\ u\left(lhw + b - k',\ R'k' + w'g(1-l,h) - b'(1+n)\right) \\ &\quad + \beta\ V\left(b',\ g(1-l,h)\right) \end{aligned}$$

(b) FOC:

$$u_1 = R'u_2 \tag{5}$$

$$u_1 hw = g_l\ \left[u_2 w' + \beta V_h(.') \varphi\right] \tag{6}$$

$$u_2\ (1+n) = \beta\ V_b(.') \tag{7}$$

Envelope:

$$V_b = u_1 \tag{8}$$

$$V_h = u_1 wl + g_h\ \left[u_2 w' + \beta\ V_h(.') \varphi\right]$$

(c) The first foc is a standard Euler equation. The third (together with $V_1 = u_1$) states that the marginal utilities of parents and children must be the same when they overlap. The perturbation is: give up $dc^o_{t+1} = -\varepsilon$ and raise the bequest by $\varepsilon/(1+n)$. The second foc equates the marginal cost of spending a bit more time in school to its marginal benefit (higher earnings when old and more human capital of the kids). The perturbation is: $dl = -\varepsilon$ with cost $u_1 wh\varepsilon$. At $t+1$ have additional human capital $g_l\varepsilon$ and eat the earnings. Also raise children's h' by $\phi g_l \varepsilon$. Without bequests, this simplifies to

$$\frac{g_1 w_{t+1}}{R_{t+1}} = w_t h_t \tag{9}$$

which requires the marginal return to human capital to equal the interest rate.

A solution is a vector $\left(c^y_t, c^o_{t+1}, l_t, k_{t+1}, b_{t+1}\right)$ that solves the 3 focs (with the V's substituted out) and the 2 budget constraints.

(d) Market clearing:

$$\begin{aligned} K_{t+1} &= N_t k_{t+1} \\ L_t &= N_t h_t l_t + N_{t-1} g(1-l_{t-1}, h_{t-1}) \\ F(K_t, L_t) + (1-\delta) K_t &= N_t c^y_t + N_{t-1} c^o_t + K_{t+1} \end{aligned}$$

A CE consists of sequences

$$\{c^y_t, c^o_t, k_t, l_t, b_t, K_t, L_t, h_t, q_t, w_t, R_t\}$$

[11 objects] that satisfy

- 5 household conditions;
- 2 firm conditions;

- 3 market clearing conditions;
- $R_{t+1} = 1 - \delta + q_{t+1}$.
- $h_{t+1} = \varphi h^o_{t+1}$.

(e) Steady state: The interest rate is determined by the Euler equation and

$$u_2 (1 + n) = \beta u_1 (.')$$

which imply

$$R\beta = (1 + n) \tag{10}$$

The other 2 equations are

$$h = g (1 - l)$$

and

$$\begin{aligned} u_1 h w &= g_1 \left[u_1 / R + \varphi \beta u_1 w l \right] \\ h &= g_1 (1 - l) \ \ \left[1/R + \varphi \beta l \right] \end{aligned}$$

The economy is dynamically efficient (cf. (10)). When n rises, it becomes more costly to achieve utility by leaving bequests. Parents save less and R rises.

3.2 Answer: Habit formation

Based on Carroll, Overland, and Weil (2000, AER).

(a) $H = U (c, h) + \psi [Ak - c] + \lambda \rho [c - h]$.

(b)

$$\begin{aligned} \partial H / \partial c &= 0 = U_c - \psi + \lambda \rho \\ \partial H / \partial k &= \theta \psi - \dot{\psi} = \psi A \\ \partial H / \partial h &= \theta \lambda - \dot{\lambda} = U_h - \lambda \rho \end{aligned}$$

(c) Solution: $c (t), k (t), h (t), \psi (t), \lambda (t)$ that satisfy: 3 foc, 2 laws of motion, and boundary conditions: k_0 and h_0 given, $\lim_{t \to \infty} e^{-\theta t} \psi_t k_t = 0$ and $\lim_{t \to \infty} e^{-\theta t} \lambda_t h_t = 0$.

(d) Balanced growth rates: $g (\psi) = \theta - A$ from the first-order condition. The other focs then require $g (\psi) = g (\lambda) = g (U_c) = -g (U_h)$. From the constraints we have $g (c) = g (h) = g (k)$. $g (U_c) = -g (c)$.