Practice Problems: Growth Algebra and Logs

Econ520. Spring 2021. Prof. Lutz Hendricks. November 10, 2020

Optional: Jones, Macroeconomics, exercises 3.1-3.5, 3.9, 3.10.

1 Growth rate calculations

1. If $y_{1950} = 100$ and $y_{2000} = 150$, what is the average growth rate of y?
2. If y grows at rate $g = 0.02$, by how much does y grow over 50 years?
3. Ethiopia's per capita income in 2000 was \$635. Compute its per capita income in 2050 for growth rates of 1%, 2%, 4%, 6% per year. (Note that the same calculations would tell you how much your retirement savings increase over your working life for different rates of return on your investments.)
4. Plot (log) per capita GDP for the following scenarios:
   (a) Growth of 5% per year for 70 years.
   (b) Growth of 2% for 70 years, followed by 7% for 20 years, followed by 5% for 28 years.
   (c) Growth of 7% for 50 years, followed by 1% for 140 years.
5. Assume $g(x) = 5\%$ and $g(y) = 2\%$. Calculate:
   (a) $g(xy)$
   (b) $g(x/y)$
   (c) $g\left(x^{1/2}y^{1/2}\right)$
   (d) $g\left(x^{-1/3}y^{1/2}\right)$

1.1 Answer:

1. $(1+g)^{50} = \frac{y_{2000}}{y_{1950}}$
2. $(1+g)^{50}$.
3. To be written.
4. Plotting log GDP:
   (a) Let's normalize log GDP at the beginning of time to 1 (log(1) = 0). After 70 years, GDP grows to $1.05^{70} = 30.4$ so that $\log(GDP)$ =3.41. So we get a straight line from (1,0) to (70,3.41).
   (b) First 70 years: log GDP goes from 1 to $log(1.02^{70}) = 1.38$. Etc. The general point: we have straight lines with slopes equals to the growth rates.

2 Investment Fees

Suppose you put away \$100,000 at age 25. At age 65 you withdraw the funds with interest. The return on investment is 5% per year.

1. How much wealth do you withdraw at age 65?
2. How does your answer change if you pay your investment advisor 1% of the portfolio's value in fees?
3. Assume that the inflation rate is 2%. What is the real value of you portfolio at age 65 (without fees)?
4. Assume that your nominal capital income is taxed each year at a flat rate of 30%. The inflation rate is 2%. What is the real value of your portfolio at age 65?
5. Now assume that the inflation rate rises to 4%. The tax is still in effect. Calculate the real value of your portfolio at age 65. Why is it less than \$100,000, even though the real rate of return (5%-4%) is positive?

2.1 Answer: Investment fees

The point of this question is that interest rate is the growth rate of your funds.

1. The funds grow to $\$100,000 \times (1.05)^{40}$ or about \$700,000.
2. Now the rate of return (growth rate of funds) is 4% and the value at age 65 is reduced to \$480,000. A big reduction for a small looking fee.
3. The real value is [nominal value]/[price index]. The price index today is 1 (a normalization). In 40 years it is 1.02^{40}. The real value is $\$100,000 \times 1.05^{40}/1.02^{40}$ or about \$320,000.
4. Capital income is 5% of the portfolio value. After tax, it is $0.05 \times (1 - 0.3)$. Same equation as #3 with the new rate of return. Terminal value about \$395,000.

3 Log scale

1. Assume that GDP (y_t) grows at a constant rate of 3% per year for 50 years and then at a constant rate of 1% per year for 20 years. Plot $\log(y_t)$ against time t. This need not be to scale.
2. How does your graph change if the growth rate for the first 50 years rises to 4% per year?

4 Growth rates

1. If $Y_t = K_t^{1/3}$ and K_t grows at 6% per year, what is the growth rate of Y_t?
2. If nominal GDP grows at 5% per year and real GDP grows at 2% per year, how much inflation is there over 20 years, i.e., calculate [GDP deflator at $t+20$] / [GDP deflator at t]. Show the steps of your calculations.
3. Imagine that $x(t)$ grows at the constant rate g.
   (a) Plot $\ln(x(t))$ against time t. Explain the key features of the graph.
   (b) Now suppose that the growth rate increases permanently, starting in year 50, to the constant rate $\hat{g} > g$. How does this change the plot of $\ln(x(t))$?
4. The production function is $Y_t = \bar{A}K_t^{1/3}L_t^{2/3}$. The growth rates of the inputs are $g(K) = 3\%$ and $g(L) = 1\%$. $\bar{A}$ does not grow. Find the growth rate of Y.
5. Given: GDP in 1990 is $1,000. GDP in 2000 is $1,344. Write down a general expression that can be used to calculate the average growth rate of a variable over several years. Calculate the average growth rate of GDP per year.

4.1 Answer: Growth rates

1. 2%. Or, more precisely, $1.06^{1/3} - 1$.
2. Inflation rate: 3% per year. Growth of prices over 20 years: $1.03^{20} = 1.8$.
3. $x(t)$ grows at rate g.
   (a) straight line with slope g.
   (b) this adds a kink at $t = 50$.
4. $g(Y) = 1/3 \times 3\% + 2/3 \times 1\% = 1.67\%$.
5. Average growth rate: $(GDP_{2000}/GDP_{1990})^{1/10} - 1 = 3\%$.