Practice Problems: Solow Model

Econ520. Fall 2021. Prof. Lutz Hendricks. October 29, 2021

Jones, Economic Growth, exercises 2.1 to 2.6.

Blanchard/Johnson, Macroeconomics, ch. 11, questions 1-4, 6-9.

Jones, Macroeconomics, exercises 5.1-5.1, 5.7, 5.10.

1 Levels accounting

1. Why does the Solow model imply a larger role for productivity in explaining cross-country income gaps than does the production model?

2. Graph the effect of a higher $\bar{A}$ on steady state output. Decompose it into a direct effect of $\bar{A}$ on y^* and an indirect effect going through k^*.

2 Properties of the Solow Model

1. Derive that law of motion for $k = K/L$ in the Solow model. What is the interpretation?

2. Show that the Solow model has a unique steady state using a diagram.

3. Show that the economy converges to the steady state from any initial $\bar{k}_0$. Show that the growth rate declines as the economy approaches the steady state.

4. Consider two countries with identical technology parameters and production function $Y = K^\alpha \ (AL)^{1-\alpha}$. The only difference is that the saving rate is $s_A = 0.05$ and $s_B = 0.2$. $\alpha = 1/3$. How large is the gap in per capita income between A and B? How does your answer change if $\alpha = 0.8$?

5. In what sense does the Solow model predict that poor countries grow faster?

6. Does the Solow model explain the growth experiences of countries over the post-war period? Why or why not?

2.1 Answer: Properties of the Solow model

1.-3. We did that in class.

4. Assume both countries are close to the steady state, so we can use the steady state income equation $y = A(s/[n+\delta])^{\alpha/(1-\alpha)}$. Take the ratio: $y_B/y_A = (s_B/s_A)^{\alpha/(1-\alpha)}$. Now plug in numbers.

5.-6. We did these calculations in class.

3 Comparative Dynamics

1. Examine the effect of a one-time **increase in the labor force** (L) in the Solow model without technical change. Plot the time paths of Y/L, wages, and interest rates. [Jones 2.2] [Answer: the same as a decrease in K or an increase in A. The reason: only $\bar{k}$ appears in the law of motion.]

2. **Income tax:** Examine the short-run and long-run effects of an income tax. Instead of receiving $wL+rK$, households receive $(1-\tau)(wL+rK)$. The government consumes the tax revenues. [Jones 2.3] [Answer: the same as a decline in s from its original value to $s(1-\tau)$].

   (a) How does your answer change if the government instead invests the tax revenues? [Answer: the same as a rise in s.]

4 Comparative statics

Derive the effects of the following events in the Solow model. Assume that the economy starts in the steady state. Plot the transition path of $k = K/L$ following the shock.

1. A permanent increase in s.

2. A permanent increase in the size of the population (not in the growth rate n).

3. A permanent increase in productivity growth, g.

5 Varying the capital share

Compare two Solow economies. Economy A has $\alpha = 1/3$. Economy B has $\alpha = 2/3$. Otherwise, the economies are identical.

1. Draw the Solow diagrams (sy_t and $(n+\delta)\,k_t$ against k_t) for both cases. Explain how they differ.

2. For both economies, assume that k_0 equals 10% of the steady state value of k. In which economy do you expect growth to be faster at date 0? Which economy do you expect to approach its steady state faster? Explain the intuition. You need not derive your answer. *Hint:* Find $\dot{k}$ in the Solow diagram for some low value of k. For which economy is it larger?

3. Derive an equation for the growth rate of k as a function of the distance from the steady state (k/k^*). How does the growth rate of k depend on α? Verify your intuition of part 2.

5.1 Answer

1. Low α: the production function has lots of curvature.

2. We know that $g(k) = A^{1-\alpha} s k^{\alpha-1} - (n+\delta)$. For small α, $k^{\alpha-1}$ gets very large when k is small. Or in the graph: with more curvature, the gap between sy and $(n+\delta)k$ is larger at low k. High MPK means that the economy can grow fast.

3. Just multiply and divide by $(k^*)^{1-\alpha}$:

$$g(k) = A^{1-\alpha} s\,(k^*)^{\alpha-1}\,(k/k^*)^{\alpha-1} - (n+\delta) \tag{1}$$

Now think about what happens to $(k/k^*)^{\alpha-1}$ for small and large α. If α is very close to 1, $(k/k^*)^{\alpha-1} \approx (k/k^*)^0 = 1$ and the growth rate does not respond much when k changes. But if α is very close to 0, $(k/k^*)^{\alpha-1} \approx (k/k^*)^{-1}$ and the growth rate responds strongly when k changes.

6 Solow Model With Subsistence

Consider a version of our Solow model where households require a subsistence level of consumption. Here are the details. The production function is

$$Y = K^\alpha \ (AL)^{1-\alpha} \tag{2}$$

and capital is accumulated according to

$$\dot{K}_t = sY_t - \delta K_t \tag{3}$$

Labor input grows at a constant rate n. Productivity is constant at A. The new part is the saving rate. While income is lower than a threshold $\bar{y}$ the household does not save. Of the income above $\bar{y}$ the household saves a constant fraction $\bar{s}$. Per capita saving is therefore given by $sy = 0$ if $y < \bar{y}$ and

$$sy = \bar{s} \ (y - \bar{y}) \tag{4}$$

if $s \geq \bar{y}$.

1. Graph the saving rate against y.
2. Plot the Solow diagram for this model. By this I mean a diagram depicting sy and $(n + \delta)\,k$. For comparison also plot the sy line without subsistence ($\bar{y} = 0$).
3. How many steady states does the model have? Assume that $\bar{y}$ is not too large – otherwise the economy has no steady states with $k > 0$.
4. For various values of initial capital, characterize which steady states the economy may converge to.

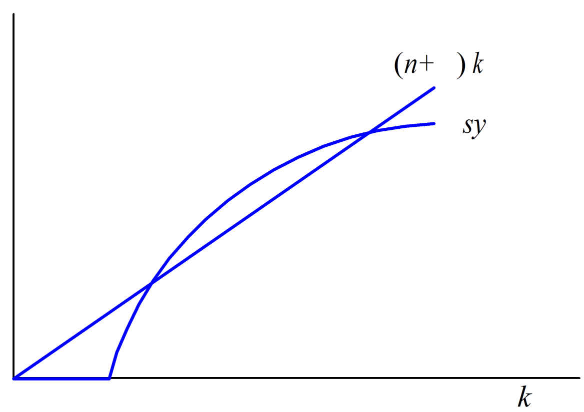

Figure 1: Solow Model with Subsistence Consumption

6.1 Answer: Solow Model with Subsistence

1. $s = \max\{0, \bar{s}(1 - \bar{y}/y)\}$. This starts at 0 until y reaches $\bar{y}$. Then it rises towards $\bar{s}$.

2. Solow diagram: Note that $sy = \bar{s}A^{1-\alpha}k^{\alpha} - \bar{s}\bar{y}$. The sy line is simply shifted down by a constant. It crosses the x-axis where $y = \bar{y}$. See figure 1.

3. The model has 3 steady states (including $k = 0$).

4. The highest steady state is similar to the regular Solow model and locally stable. The steady state at $k = 0$ is also stable. The middle steady state is unstable. The model has a poverty trap.

7 Convergence

7.1 Galton's Fallacy

[Based on Jones 3.4]

1. Imagine you have data on the height of mothers (h_m) and daughters (h_d). You plot the change in daughter's height ($h_d - h_m$) against

mother's height and find a negative relationship. Tall mothers tend to have daughters that are less tall than their mothers. Does this imply that all persons converge over time to the same height? Why or why not?

2. Now imagine that mother's height and daughter's height are drawn independently from the same distributions. What will you find if you plot the change in height against mother's height?

3. What does all this imply for the interpretation of cross-country growth regressions that find a negative relationship between initial income and subsequent income growth?

7.1.1 Answer: Galton's Fallacy

1. Heights need not converge. Counter-example: each person draws her height from a fixed distribution, independently of mother's height. The reason why the change in height is small when the mother is tall is simply that everyone has (on average) average height daughters. Tall mothers therefore have daughters that are shorter than themselves.

2. See #1.

3. In some samples of countries we find that $g(y)$ is negatively related to y at the beginning of the sample. This does not imply σ convergence.