# Stochastic Dynamic Programming: Theorems

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# Theorems: Stochastic DP

We state conditions under which dynamic programming "works"

The assumptions needed and the results are very similar to the deterministic case.

Acemoglu (2009) has a simplified version with discrete random variables.

Stokey et al. (1989) have more general results.

# The generic problem
## Environment

Start with capital stock $x(0)$.

The shocks $z(t)$ follow a discrete Markov chain.

- A strong assumption.
- It can be relaxed without affecting results too much.
- But at the expense of notation.

# The generic problem

After each node $z^t = (z(0), ..., z(t))$ of shocks, choose next period's capital stock

$$x(t+1) = x[z^t] \tag{1}$$

$x(t+1)$ is constrained to lie in the set $G(x(t), z(t))$.

Key: period utility and constraints only depend on current realizations $(x(t), z(t))$, not on history.

If not: DP fails (or we need tricks).

# P1: Sequence problem

$$V^*(x(0), z(0)) = \max_{\{x[z^t]\}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(x\left[z^{t-1}\right], x[z^t], z(t)\right)$$

subject to

$$\begin{aligned} x[z^t] \quad &\in \quad G\left(x\left[z^{t-1}\right], z(t)\right) \\ &x(0) \text{ given} \end{aligned}$$

The law of motion for $x$ is built into $U$.

# P2: Recursive problem

$$V(x,z) = \max_{y \in G(x,z)} U(x,y,z) + \beta E\left[V\left(y,z'\right)|z\right]$$

So much simpler!

# Assumptions

A1:

- $G(x, z)$ is nonempty.
- For all feasible plans:
  $\lim_{n\to\infty} E\left[\sum_{t=0}^{\infty} \beta^t U\left(x\left[z^{t-1}\right], x[z^t], z(t)\right) | z(0)\right] < \infty$.

A **feasible plan** is now a collection of state contingent plans ($x[z^t]$ for all histories $z^t$) that satisfies $x[z^t] \in G$.

# A2

- $X$ is a compact subset of $\mathbb{R}^K$
  - where $x(t) \in X$.
- $U$ is continuous.

[There are additional issues when $z$ does not live in a finite set]

# Theorem 1: Equivalence of values

- Assume A1 and A2.
- $V^*(x,z)$ in the sequence problem solves the recursive problem.
- $V(x,z)$ in the recursive problem equals $V^*(x,z)$ in the sequence problem, IF

$$\lim_{t\to\infty} \beta^t E\left[V\left(x\left[z^{t-1}\right], z(t)\right)\right] = 0$$

  for all feasible plans.

# Theorem 2: Principle of Optimality

- Assume A1 and A2.
- Any optimal plan in the sequence problem satisfies the Bellman equation with value $V^*$:

$$V^*\left(x\left[z^{t-1}\right], z(t)\right) = U\left(x\left[z^{t-1}\right], x[z^t], z(t)\right) + \beta E[V^*(x[z^t], z(t+1)) | z($$

- Any feasible plan that solves the above attains $V^*$ in the sequence problem.

The point: Solving P1 and P2 are equivalent.

# Theorem 3: Existence of solutions

- Assume A1 and A2.
- An optimal plan exists for any initial conditions $x(0), z(0)$.
- $V$ is unique, continuous, bounded in $x$ for each $z$.

# A3

- $U(x,y,z)$ is strictly concave in the sense that

$$U(\bar{x},\bar{y},z) \geq \alpha U(x,y,z) + (1-\alpha) U(x',y',z) \qquad (2)$$

  with strict inequality when $x \neq x'$, where $\bar{x} = \alpha x + (1-\alpha) x'$ and $\bar{y} = \alpha y + (1-\alpha) y'$.
- The set $G(x,z)$ is convex in the sense that

$$y \in G(x), y' \in G(x') \Longrightarrow \bar{y} \in G(\bar{x}) \qquad (3)$$

# A4

- $U(x,y,z)$ is strictly increasing in all elements of $x$.
- $G$ is monotone: $x \leq x' \Longrightarrow G(x,z) \subset G(x',z)$ for all $z$.

# A5

- $U$ is continuously differentiable in $x$.

# Theorem 4: Concavity of V

- Assume A1, A2, A3.
- Then $V$ is strictly concave in $x$.
- The optimal plan $x[z^t] = \pi(x(t), z(t))$ is unique and $\pi$ is continuous in $x$.

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Recall A3: $U$ strictly concave. $G$ convex.

Then the Bellman equation is a concave optimization problem.

# Theorem 5: Monotonicity of V

- Assume A1, A2, A4.
- Then $V$ is strictly increasing in $x$.

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Recall A4: $U$ and $G$ are monotone in $x$.

# Theorem 6: Differentiability of V

- Assume A1-A3, A5.
- Then $V$ is continuously differentiable in $x$.

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Recall A5: $U$ is differentiable.

# Theorem 8: Euler equations

- Assume A1-A5.
- Then the interior feasible plan that satisfies the Euler equation

$$D_y U(x, \pi(x,z), z) + \beta E\left[D_x U\left(\pi(x,z), \pi\left(\pi(x,z), z'\right), z'\right) | z\right] = 0$$

  and the TVC

$$\lim_{t\to\infty} \beta^t E\left[D_x U(t)\ \ \pi(t)\right] = 0$$

  solves the recursive problem P2.
- The Euler equation with scalar $x$:

$$\frac{\partial U(x, x', z)}{\partial x'} + \beta E\left[\frac{\partial U(x', x'', z')}{\partial x'} | z\right] = 0 \qquad (4)$$

- Note: the TVC must hold starting from any node $x[z^t], z(t)$.

# Continuous shocks

- If the shock $z$ lives in a continuum, nothing of substance changes.
- Acemoglu 16.4

# Example: Permanent income hypothesis

# Example: Permanent income hypothesis

- A household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c(t)) \tag{5}$$

- $u$ has all nice properties: strictly increasing, concave, differentiable.
- Budget constraint:

$$a(t+1) = (1+r)a(t) + w(t) - c(t) \tag{6}$$

- $w(t)$ is i.i.d. with $\Pr(w(t) = w_j) = q_j$.

# Budget constraint

- A tricky issue: the budget constraint.
- How much should the household be able to borrow?
- A natural borrowing constraint:

$$a(t) \geq -\sum_{s=0}^{\infty} \frac{w_1}{(1+r)^s} \equiv -b_1 \tag{7}$$

- This ensures that the household can repay his debts, even if he receives the worst possible income realization in each period.
- Because $w$ is i.i.d. and the household lives forever, $b_1$ is a constant.

# Sequence problem

- The history is $w^t$.
- The household chooses $c\left[w^t\right]$ for all possible histories.
- The problem is too tedious to even write down...

# Recursive formulation

$$V(a,w) = \max_{a' \in [-b_1,(1+r)a+w]} u\left([1+r]a+w-a'\right) + \beta EV\left(a',w'\right) \quad (8)$$

Mapping into the generic problem:

- $x \rightarrow a, z \rightarrow w$
- $G(z,x) \rightarrow [-b_1,(1+r)a+w]$
- $U(x,y,z) \rightarrow u([1+r]a+w-a')$

# First-order conditions

Verify A1-A5 ...

Then we can characterize the solution by the FOCs:

$$u'(c) = \beta E V_a(a', w') \tag{9}$$

$$V_a(a, w) = (1+r) u'(c) \tag{10}$$

Euler:

$$u'(c) = \beta (1+r) E u'(c') \tag{11}$$

# Verify A1

- $G(x,z) = [-b_1, (1+r)a + w]$ is nonempty – $b_1$ is constructed that way.
- For all feasible plans: $\lim_{t\to\infty} E\left[\sum_{t=0}^{\infty} \beta^t U\left(x\left[z^{t-1}\right], x[z^t], z(t)\right) | z(0)\right] < \infty$.
  - This is NOT generally satisfied.
  - $(1+r) > \beta$ could imply unbounded growth.
  - We need a restriction that $r$ not too high. Tedious details...

# Verify A2

- $X$ is a compact subset of $\mathbb{R}^K$
  - Here: $X = \mathbb{R}_+$ which is obviously not compact.
  - We need to argue that bounding $a$ from above does not bind (when $1 + r < \beta$).
- $U$ is continuous – by assumption.

# Verify A3

- $U(x, y, z)$ is strictly concave in the sense that

$$U(\bar{x}, \bar{y}, z) \geq \alpha U(x, y, z) + (1-\alpha) U\left(x', y', z\right) \tag{12}$$

  with strict inequality when $x \neq x'$, where $\bar{x} = \alpha x + (1-\alpha) x'$ and $\bar{y} = \alpha y + (1-\alpha) y'$.

  - Here: $U([1+r][\alpha a_1 + (1-\alpha) a_2] + w - [\alpha a_1' + (1-\alpha) a_2'])$ with $\partial U / \partial a' < 0$ and $\partial^2 U / \partial (a')^2 < 0$.

- The set $G(x, z)$ is convex in the sense that

$$y \in G(x), y' \in G\left(x'\right) \Longrightarrow \bar{y} \in G(\bar{x}) \tag{13}$$

  - easy to check

# Verify A4

- $U(x, y, z)$ is strictly increasing in all elements of $x$.
  - Here: $\partial u / \partial a > 0$.
- $G$ is monotone: $x \leq x' \Longrightarrow G(x, z) \subset G(x', z)$ for all $z$.
  - Here: $(1+r)a$ is increasing in $a$.

# Verify A5

- $U$ is continuously differentiable in $x$. – by assumption.

# Quadratic case

- Assume $u(c) = \phi c - 0.5c^2$
- $u'(c) = \phi - c$.
- Euler:
$$\phi - c = \beta (1+r) E [\phi - c'] \tag{14}$$
- Nothing in the info set at $t$ should predict consumption growth [in this example: $Ec' - c$]. Hall 1978.
- Strangely, a large literature has tested this prediction, even though it only holds with quadratic utility!

# Shortcut

- We could have derived the Euler equation naively by treating $E$ as a constant in the optimization problem.
- The deterministic FOCs turn out to be correct (in many [all?] cases).

# Reading

- Acemoglu (2009), ch. 16.1-16.2.
- Krusell (2014), ch. 6.

# References I


Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .