Stochastic Growth Model

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Introduction

We now return to the stochastic growth model. We study

- the planner’s problem
- the competitive equilibrium

Then we introduce heterogeneity and risk sharing.
Planning solution

The history of shocks is $\theta^t$.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t|\theta_0) u(c[\theta^t])$$  \hspace{1cm} (1)$$

Technology:

$$X = F(K,L,\theta) + (1 - \delta)K - c$$  \hspace{1cm} (2)$$

$$K' = X$$  \hspace{1cm} (3)$$
Define $k = K/L$.

$$V(k, \theta) = \max_{k' \in [0, f(k, \theta) + (1 - \delta)k]} u\left(f(k, \theta) + (1 - \delta)k - k'\right)$$  \hspace{1cm} (4)$$

$$+ \beta E \left[V\left(k', \theta'\right) | \theta\right]$$  \hspace{1cm} (5)$$
First-order conditions

- Verify that A1-A5 hold ... Theorems 1-6 apply.

- FOC

\[ u'(c) = \beta E V_k (k', \theta') \]

- Envelope

\[ V_k (k, \theta) = u'(c) [f_k (k, \theta) + 1 - \delta] \]

- Euler

\[ u'(c) = \beta E [u' (c')] \{f_k (k', \theta') + 1 - \delta\} | \theta \] \quad (6)

- Solution: \( V(k, \theta) \) and \( \pi(k, \theta) \) that "solve" the Bellman equation
Characterization

- Now for the bad news ... there really isn’t much one can say about the solution analytically.
- But see Campbell (1994) for a discussion of a log-linear approximation.
Competitive Equilibrium
The model comes in 2 flavors.

1. Complete markets
   - for every history, there exists an asset that pays in that state of the world
   - the implication is complete risk sharing: all idiosyncratic risks are insured
   - aggregate risks remain

2. Incomplete markets
   - some securities are missing
   - there is no representative agent
Trading arrangements

- With complete markets, date 1 Arrow-Debreu trading is convenient
  - Uncertainty essentially disappears from the model.
- With incomplete markets, it is easiest to specify the set of securities available at each date.
  - Sequential trading.
Complete markets - Arrow Debreu trading

- The environment is standard.
- The history is of shocks is $\theta^t$.
- Trading takes place at date 1.
- The point: This looks like a static model without uncertainty.
Market arrangements

Goods markets: standard

- buy and sell consumption at each node $\theta^t$
- price $p(\theta^t)$

Labor markets: standard

- wage $w(\theta^t)$

Capital rental:

- households can buy goods in $\theta^t$ and give them to firms
- firms then pay $R(\theta^{t+1})$ tomorrow
- this includes returning the undepreciated capital
Expenditures in state $\theta^t$:

$$x(\theta^t) = p(\theta^t)[c(\theta^t) + s(\theta^t)]$$  \hspace{1cm} (7)

$p(\theta^t)$ is the price of the good in state $\theta^t$.
$c$ is consumption
$s$ is "saving:" buy goods (capital) and rent to firms.
Household: budget constraint

Income in state $\theta^t$: 

$$y(\theta^t) = w(\theta^t) + R(\theta^t)s(\theta^{t-1})$$ \hspace{1cm} (8)

$w(\theta^t)$ is the wage.

$R(\theta^t)$ is the payoff from renting a unit of the good to the firm.

Both are state contingent.

Poor notation: keep in mind that $\theta^t$ follows $\theta^{t-1}$
Household: budget constraint

Lifetime budget constraint:

$$\sum_{t=0}^{\infty} \sum_{\theta^t} [y(\theta^t) - x(\theta^t)] + p(\theta_0)s_0 = 0$$  \hspace{1cm} (9)

$s_0$ is the initial endowment of goods.

With Arrow-Debreu trading, there is a lifetime budget constraint, even under uncertainty.

- Because there really is no uncertainty any more.
- At each node, the household’s spending and income are fully predictable.
Firms

Firms maximize the total value of profits.

- There is no discounting because of Arrow-Debreu trading.

Profits in state \( \theta^t \):

\[
p(\theta^t)[F(K[\theta^t],L[\theta^t],\theta_t) + (1 - \delta)K[\theta^t]]
- R(\theta^t)K(\theta^t) - w(\theta^t)L(\theta^t)
\]

Value of the firm: sum of profits over all states.

FOCs are standard:

- since the firm does not own anything, it maximizes profits state-by-state.
Competitive Equilibrium

- Allocation: $c(\theta^t), s(\theta^t), K(\theta^t), L(\theta^t)$.
- Price system: $p(\theta^t), w(\theta^t), R(\theta^t)$ for all histories $\theta^t$.
- These satisfy:
  1. Household optimality.
  2. Firm optimality.
  3. Market clearing:
     - $L(\theta^t) = 1$.
     - $K(\theta^t, \theta_{t+1}) = s(\theta^t)$.
     - Goods market.
This looks like a static model without uncertainty.
  ▶ Each history defines new goods: output, labor, capital rental.
▶ The setup is far more complicated than the recursive one.
Risk Sharing

- What if agents are heterogeneous?
- With complete markets, risk is perfectly shared.
- The simplest case: An endowment economy with Arrow-Debreu trading.
- The state is $\theta^t$. 
There are $I$ types of households, indexed by $i$.

Endowments are $y^i(\theta^t)$.

Preferences are

$$\sum_t \sum_{\theta^t} \beta^t q(\theta^t) u^i(c^i[\theta^t])$$

Budget constraints:

$$\sum_t \sum_{\theta^t} p(\theta^t) [c^i(\theta^t) - y^i(\theta^t)] = 0 \quad (10)$$
First-order conditions are as usual:

\[ q(\theta^t) \beta^t \frac{\partial u^i(c^i[\theta^t])}{\partial c^i[\theta^t]} = \lambda_i p(\theta^t) \]  

where \( \lambda_i \) is the Lagrange multiplier.
Risk Sharing

Complete risk sharing: For all \( \theta^t \) the MRS is equated across households:

\[
MRS \left( \theta^t, \hat{\theta}^\tau \right) = -\frac{\beta^t \frac{\partial u^i (c^i [\theta^t])}{\partial c^i [\theta^t]}}{\beta^\tau \frac{\partial u^i (c^i [\hat{\theta}^\tau])}{\partial c^i [\hat{\theta}^\tau]}} = \frac{p(\theta^t)/q(\theta^t)}{p(\hat{\theta}^\tau)/q(\hat{\theta}^\tau)}
\]

Equivalently, the ratio of marginal utilities between 2 agents is the same for all \( \theta^t \):

\[
\frac{\partial u^i (c^i [\theta^t])}{\partial c^i [\theta^t]} / \frac{\partial u^j (c^j [\theta^t])}{\partial c^j [\theta^t]} = \frac{\lambda_i}{\lambda_j}
\] (12)
Implications

Individual consumption still fluctuates because the aggregate endowment changes over time.

- aggregate risk cannot be insured

If there is no aggregate uncertainty, then individual consumption is constant.

Proof:

\[
\frac{\partial u^i}{\partial c^i} = \left(\frac{\lambda_i}{\lambda_1}\right) \frac{\partial u^1}{\partial c^1}
\]  

(13)

That implies an increasing function \( c^i = f_i(c^1) \) that is the same for all states \( \theta^t \).

Market clearing: \( \sum_i c^i = \sum_i f_i(c^1) = y \).

This has a unique solution \( c^1 \). □
Sequential Trading
We set up the C.E. with sequential trading.

If we want complete markets, we need **Arrow securities**.

Each security, \( a(\theta^{t+1}) \) is indexed by the state of the world in which it pays off: \( \theta^{t+1} \).

The asset is purchased for price \( \bar{p}(\theta^t, \theta') \) in state \( \theta^t \).

It pays one unit of consumption if \( \theta^{t+1} = [\theta^t, \theta'] \).
Household

- **Budget constraint:**
  
  \[
  c(\theta^t) + s(\theta^t) = w(\theta^t) + a(\theta^t) + R(\theta^t)k(\theta^t) \quad (14)
  \]

  \[
  s(\theta^t) = \sum_{\theta_{t+1}} \bar{p}(\theta^t, \theta_{t+1}) a(\theta^t, \theta_{t+1}) + x(\theta^t) \quad (15)
  \]

  \[
  k(\theta^t, \theta_{t+1}) = x(\theta^t) \quad (16)
  \]

- **Numeraire:** consumption at each node \(\theta^t\).
Household problem:

\[
\max \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t|\theta_0) u(c[\theta^t])
\]  

s.t. budget constraints for all \(\theta^t\).
Recursive household problem

- State: \((\vec{a}, k, \theta)\).
  - \(\vec{a}\): holdings of all the \(a(\theta)\).
- Given prices: \(w\) and \(\bar{p}(\theta, \theta')\).
- Bellman equation:

\[
V(\vec{a}, k, \theta) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta' | \theta) \, V(\vec{a}', k', \theta')
\]

s.t. budget constraint

\[
\sum_{\theta'} \bar{p}(\theta, \theta') \, a'(\theta') + k' + c = w + a(\theta) + Rk
\]
First order conditions

- For $\alpha'(\theta')$:

$$ u'(c)\bar{p}(\theta, \theta') = \beta q(\theta' | \theta) \frac{\partial V(\overline{\alpha'}[\theta'], k', \theta')}{\partial a(\theta')} $$  \hspace{1cm} (18) $$

- For $k'$:

$$ u'(c) = \beta \sum_{\theta'} q(\theta' | \theta) \frac{\partial V(\overline{\alpha'}, k', \theta')}{\partial k'} $$  \hspace{1cm} (19) $$
First order conditions

- Envelope:
  \[
  \frac{\partial V(\hat{a}, k, \theta)}{\partial a(\theta)} = u'(c) \quad (20)
  \]
  \[
  \frac{\partial V(\hat{a}, k, \hat{\theta})}{\partial a(\theta)} = 0 \quad (21)
  \]
  \[
  \frac{\partial V(\hat{a}, k, \theta)}{\partial k} = u'(c)R \quad (22)
  \]

- Euler equation holds state by state for state contingent claims:
  \[
  u'(c) \bar{p}(\theta, \theta') = \beta q(\theta'|\theta) \ u'(c \ [a'(\theta'), \theta']) \quad (23)
  \]

- Euler equation for capital:
  \[
  u'(c) = \beta \sum_{\theta'} q(\theta'|\theta) R(\theta, \theta') \ u'(c \ [a'(\theta'), k', \theta']) \quad (24)
  \]
  \[
  = \beta E \ R' \ u'(c')
  \]
No arbitrage

Since capital can be replicated by buying a set of Arrow securities:

$$\sum_{\theta'} \bar{p}(\theta, \theta') R(\theta, \theta') = 1$$  \hspace{1cm} (25)

Proof: Solve (23) for $q(\theta'|\theta)$ and substitute into (24).
We can write down a sequential equilibrium definition, similar to the Arrow-Debreu.

- Everything is indexed by $\theta^t$.

More powerful: Recursive Competitive Equilibrium.

- Everything is a function of the current state.
Recursive CE

- Define an aggregate state vector: \( S = (\theta, K) \).
  - In general: we need to keep track of the distribution of \((\theta_i, k_i)\) across households.
  - Here: all households are identical.

- The law of motion for the aggregate state:
  \[
  \Pr(\theta' | \theta) = q(\theta' | \theta) \\
  K' = G(\theta, K)
  \]

  where \( G \) is endogenous.
Recursive CE

Household

Given:
- aggregate state and its law of motion.
- price functions: $w(S), R(S)$ and $\bar{p}(S, \theta')$.

Bellman equation:

$$V(\vec{d}, k, S) = \max_{c, \vec{a}'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta' | \theta) V(\vec{d}' [\theta'], k', S')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w(S) + a(\theta) + R(S) k$$

and aggregate law of motion

$$S' = G(S)$$
Recursive CE

- First-order conditions: unchanged.
- Solution: $V(a,k,S)$ and policy functions $c(a,k,S)$,
  $k' = \kappa(a,k,S)$. 
Recursive CE
Firm

- Always the same because the firm has a static problem:
- Solution: $R(S), w(S)$. 
Recursive CE

- **Equilibrium objects:**
  1. Household: Value function and policy functions.
  3. Aggregate law of motion: \( K' = G(\theta, K) \).

- **Equilibrium conditions:**
  1. Household optimality.
  2. Firm optimality.
  4. Consistency:

\[
G(\theta, K) = \kappa(K, \theta, K) \tag{26}
\]

where the household’s policy function is \( k' = \kappa(k, \theta, K) \).
Note: We could toss out all the Arrow securities without changing anything.

The model boils down to:

1. Euler equation for $K$: $u'(c) = \beta E[R' u'(c')]$  
2. Law of motion for $K$: $K' = F(K, L) + (1 - \delta) K - c.$  
3. FOC: $R = F_K(K, L) + 1 - \delta.$

This changes when individuals are not identical.
Recursive CE

What do we gain?

- Avoid having to carry around infinite histories.
- Equilibrium contains few objects.
  - Especially when the economy is stationary.
- All endogenous objects are functions.
  - Results from functional analysis can be used to determine their properties.
- Recursive CE is easy to compute.
Reading

- Acemoglu (2009) ch. 16-17.
- Krusell (2014) ch. 6
- Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains. Ch. 7 covers complete market economies (Arrow-Debreu and sequential trading). Ch. 6: Recursive CE.
- Campbell (1994) discusses an analytical solution (approximate)


