Asset Pricing

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Topics

1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?

We use the Lucas (1978) fruit tree model.

- The implications are far more general than the simple model.
- The model forms the basis for the CAPM and the β risk measure.

2. The Lucas (1978) Fruit Tree Model

Demographics:

- A single representative household.

Preferences:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

E_0 is the expectation as of time $t = 0$.

Technology

- This is an endowment economy.
- There are K identical fruit trees.
- Each tree yields d_t units of consumption goods in period t.
- d_t is random and the same for all trees.
- Trees cannot be produced.
- Fruits cannot be stored.

Technology

- The aggregate resource constraint:

$$c_t = Kd_t \tag{2}$$

- Assume that d is a finite Markov chain with transition matrix $\pi(d', d)$.
- An important feature: All uncertainty is **aggregate**.
- There are no opportunities for households to insure each other.
- This is why we can work with a representative household.

Markets

- There are markets for fruits and for trees.
- There is also a one period bond, issued by households (in zero net supply).
  - Its purpose is to determine a risk-free interest rate.
- Digression: Why this is a good model...

2.1. Household problem

- The household starts out with bonds (b_0) and shares (k_0).
- At each date, he chooses c_t, b_{t+1}, k_{t+1}.
- The **budget constraint** is

$$p_t k_{t+1} + b_{t+1} = R_t b_t + (p_t + d_t) k_t - c_t \tag{3}$$

- Notation:
  - p : the price of trees. Suppressing dependence on the state.
  - R : the real interest rate on bonds.
  - the price of bonds is normalized to 1 (how?).

Household problem

$$V(k,b,d) = \max u(c) + \beta EV\left(k',b',d'\right) \tag{4}$$

subject to

$$Rb + (p+d)k - c + pk' - b' = 0 \tag{5}$$

Household problem

First-order conditions:

$$
\begin{aligned}
c &: \quad u'(c) = \lambda \\
k' &: \quad \lambda p = EV_k(k', b', d') \\
b' &: \quad \lambda = EV_b\left(k', b', d'\right)
\end{aligned}
$$

Envelope:

$$
\begin{aligned}
V_k &= \lambda (p + d) \\
V_b &= \lambda R
\end{aligned}
$$

Euler equations

$$
\begin{aligned}
u'(c_t) &= \beta E_t \left\{ u'(c_{t+1}) R_{t+1} \right\} \\
&= \beta E_t \{ u'(c_{t+1}) \underbrace{\frac{p_{t+1} + d_{t+1}}{p_t}}_{R^S_{t+1}} \}
\end{aligned}
$$

This is very general - holds for any number of assets / for any type of asset.

Question: Why doesn't the correlation of asset returns show up anywhere?

Solution

- A solution consists of state contingent plans $\{c(d^t), k(d^t), b(d^t)\}$ for all histories d^t.
- These satisfy:
  - 2 Euler equations
  - 1 budget constraint.
  - b_0 and k_0 given.
  - Transversality: $\lim_{t\to\infty} E_0 \beta^t u'(c_t)\ [b_t + p_t k_t] = 0$.

2.2. Equilibrium: Market clearing

For every history we need:

Bonds:

$$b_t = 0$$

Trees:

$$k_t = K_t$$

Goods:

$$c_t = K_t d_t$$

There is no trade in equilibrium!

Competitive Equilibrium

- A CE consists of:
  1. an allocation: $\{c(d^t), b(d^t), k(d^t)\}$.
  2. a price system: $\{p(d^t), R(d^t)\}$
- These satisfy:
  1. household: 2 Euler equations and 1 budget constraint.
  2. 3 market clearing conditions.

Recursive Competitive Equilibrium

Objects:

- Solution to the household problem: $V(k,b,d)$ and $c(k,b,d)$, $k' = \kappa(k,b,d)$, $b' = B(k,b,d)$.
- Price functions: $p(d), R(d)$.

Equilibrium conditions:

- Household: 4
- Market clearing: 3
- No need for consistency: law of motion of the aggregate state is exogenous.

2.3. Consumption smoothing

The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \tag{6}$$

Define: Marginal rate of substitution:

$$MRS_{t+1} = \beta\, u'(c_{t+1})/u'(c_t) \tag{7}$$

MRS_{t+1} is inversely related to consumption growth.

Consumption smoothing

With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$:

$$\begin{aligned} u'(c) &= c^{-\sigma} && (8) \\ MRS_{t+1} &= \beta\ (c_{t+1}/c_t)^{-\sigma} && (9) \end{aligned}$$

Euler equation:

$$\beta \mathbb{E}_t \left\{ (1 + g(c_{t+1}))^{-\sigma} R_{t+1} \right\} = 1 \qquad (10)$$

High interest rate

$\Longrightarrow$ reward for postponing consumption

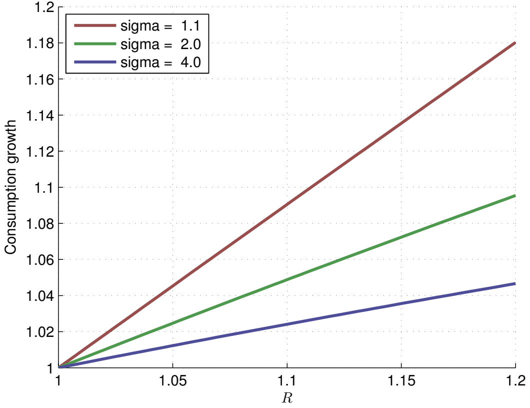
$\Longrightarrow$ high consumption growth

The **coefficient of relative risk aversion** (σ) determines how much consumption growth responds to interest rates.

Consumption smoothing

High σ implies that the household chooses smooth consumption.

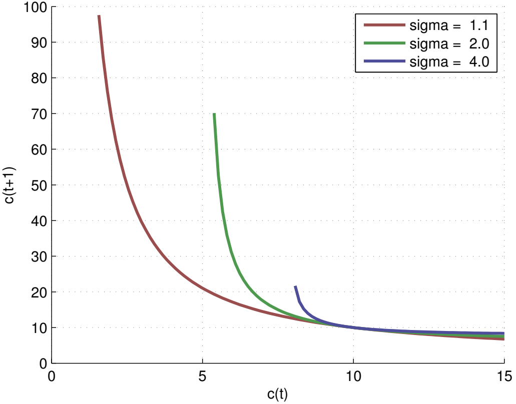
Illustration for the deterministic case:



Consumption smoothing

With high σ, marginal utility changes a lot when c changes.

The household then keeps c smooth.



3. Asset Prices

Asset pricing implications

- We will now derive the famous **Lucas asset pricing equation**.
- Define: Rate of return on trees: $R^S_{t+1} = (p_{t+1} + d_{t+1})/p_t$.
- Directly from the 2 Euler equations:

$$E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}R_{t+1}\right\} = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}R^S_{t+1}\right\} = 1$$

- Or

$$E\{MRS_{t+1}\,R_{t+1}\} = E\left\{MRS_{t+1}\,R^S_{t+1}\right\} = 1 \qquad (11)$$

When does an asset pay a high expected return?

Re-write asset pricing equation using

$$Cov(x,y) = E(xy) - E(x)E(y)$$

as

$$1 = E\{MRS\}\,E\{R\} + Cov(MRS,R) \qquad (12)$$

$$E(R) = \frac{1 - Cov(MRS,R)}{E(MRS)} \qquad (13)$$

When do assets pay high returns?

$$\mathbb{E}(R) = \frac{1 - Cov(MRS, R)}{\mathbb{E}(MRS)} \quad (14)$$

- Take a "safe" asset with fixed R.
  - $Cov(MRS, R) = 0$
  - $\mathbb{E}(R) = 1/\mathbb{E}(MRS)$.
- If $Cov(MRS, R) < 0$: the asset pays higher return than the safe asset
  - a **risk premium**
- If $Cov(MRS, R) > 0$: the asset pays **lower** return than the safe asset
  - important point: an asset return can have lots of volatility, but pay a lower return than a t-bill
  - examples?

When do assets pay high returns?

High returns require low / negative $Cov(MRS, R)$.

Example: log utility

- $u'(c) = 1/c$
- $MRS = \beta u'(c_{t+1})/u'(c_t) = \beta c_t / c_{t+1}$.

High MRS means low consumption growth.

Key implication

Assets are **risky** if their returns are **positively correlated with consumption growth**.
Then they have high **expected returns**.

Note: Uncertainty about R by itself is not priced.

- Only the part that is correlated with consumption growth is priced.

Intuition

- Imagine there are good times (high c) and bad times (low c).
- There are 2 assets: A pays dividends in good times, B pays in bad times.
- The value of the dividend is $u'(c)$.
- Assets that pay in good times are not valuable: $u'(c)$ is low.
- Assets that pay in bad times provide insurance - they are valuable (have low expected returns).

Risk (premia)

- The "risk free" assets has expected return

$$E(R_f) = \frac{1}{E(MRS)} \tag{15}$$

- A "risky" asset has expected return

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \tag{16}$$

- The **risk premium** is

$$E(R) - E(R_f) = -\frac{Cov(MRS, R)}{E(MRS)} \tag{17}$$

- This defines what **risk** means: **covariance with consumption growth**.
- Note that risk can be **negative** (insurance).

4. The Equity Premium Puzzle

The Equity Premium Puzzle

- Mehra and Prescott (1985): Asset return data pose a puzzle for the theory.
- The equity premium is "high" (6-7% p.a.)
- The cov of c growth and R_s is low.
  - The reason: Consumption is very smooth.

The Equity Premium Puzzle

TABLE 1
SUMMARY STATISTICS
UNITED STATES ANNUAL DATA, 1889–1978

	Sample Means		
R_t^s	0.070		
R_t^b	0.010		
C_t/C_{t-1}	0.018		
	Sample Variance-Covariance		
	R_t^s	R_t^b	C_t/C_{t-1}
R_t^s	0.0274	0.00104	0.00219
R_t^b	0.00104	0.00308	–0.000193
C_t/C_{t-1}	0.00219	–0.000193	0.00127

The Equity Premium Puzzle

A back-of-the envelope calculation with CRRA utility:

$$EP = -\frac{Cov\left(\beta\left[c_{t+1}/c_t\right]^{-\sigma},\ R_s\right)}{E\left\{\beta\left[c_{t+1}/c_t\right]^{-\sigma}\right\}} \tag{18}$$

Take log utility: $\sigma = 1$.

- $Cov(MRS, R_s) \simeq -0.0022$.
- $E(MRS) \simeq 1$.
- $EP \simeq 0.2\%$.
- Replicating the observed equity premium requires **very high risk aversion** $(\sigma = 40)$.

How severe is the puzzle?

Investors forego very large returns.

Table 3

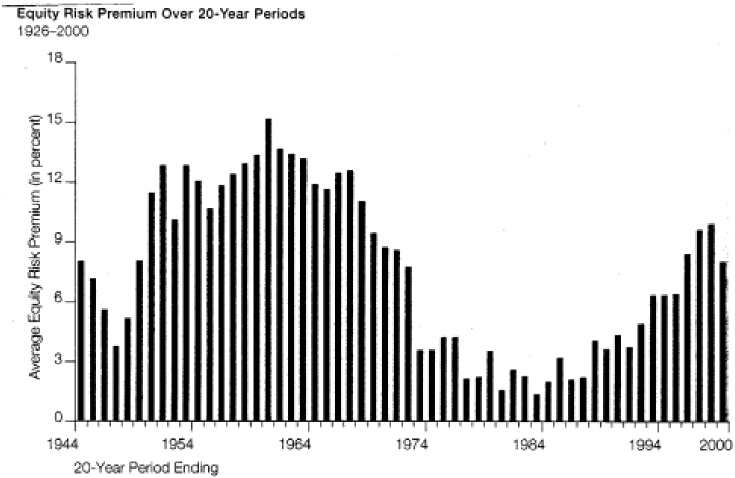
Terminal value of $1 invested in Stocks and Bonds

Investment Period	**Stocks**		**T-bills**	
	Real	Nominal	Real	Nominal
1802-1997	$558,945	$7,470,000	$276	$3,679
1926-2000	$266.47	$2,586.52	$1.71	$16.56

Source: Mehra and Prescott (2003)

Long holding periods

Over 20 year holding periods: stocks dominate bonds.



Source: Mehra and Prescott (2003)

Why do we care?

- The EP puzzle shows that we do not understand
  1. what households view as "risky"
  2. why households place a high value on smooth consumption
- This has implications for:
  1. The welfare costs of business cycles
     - They are very low in standard models.
  2. Stock price volatility.
     - Standard models fail to explain it (see below).

How to resolve the puzzle

Proposed explanations include:

1. Habit formation: $u(c_t, c_{t-1}) = \frac{[c_t - \gamma c_{t-1}]^{1-\sigma}}{1-\sigma}$.
   - Implies high risk aversion when c_t is close to c_{t-1}.
2. Heterogeneous agents
   - Implicit in the standard model: all idiosyncratic risk is perfectly insured.
3. Borrowing constraints
   - The young should hold stocks (long horizon), but cannot.
   - The old receive mostly capital income and find stocks risky.
4. Taxes / regulations (McGrattan and Prescott, 2000)
   - The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.

4.1. Beta

Now we derive the famous "beta" measure of risk.

Suppose asset m (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma \, R_{m,t+1} \tag{19}$$

The market's expected return is

$$E \; R_m - R = -\frac{Cov(MRS, R_m)}{E(MRS)} \tag{20}$$

Beta

Now we relate the covariance term to marginal utility:

$$
\begin{aligned}
Cov(MRS, R_m) &= Cov\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}, R_{m,t+1}\right) = \beta \frac{Cov(u'(c_{t+1}), R_{m,t+1})}{u'(c_t)} \\
E(MRS) &= \beta \frac{E(u'(c_{t+1}))}{u'(c_t)}
\end{aligned}
$$

Therefore:

$$
E(R_m) - R = -\frac{Cov(u'(c_{t+1}), R_{m,t+1})}{E\ u'(c_{t+1})} = \frac{\gamma\ Var(R_{m,t+1})}{E\ u'(c_{t+1})}
$$

Beta

For any asset i:

$$E\ R_i - R = -\frac{Cov\left(u'\left(c_{t+1}\right), R_i\right)}{E\ u'\left(c_{t+1}\right)} = \frac{\gamma\ Cov\left(R_m, R_i\right)}{E\ u'\left(c_{t+1}\right)}$$

Take the ratio for assets i and m:

$$\beta_i = \frac{\mathbb{E}R_i - R}{\mathbb{E}R_m - R} = \frac{Cov\left(R_m, R_i\right)}{Var\left(R_m\right)} \tag{21}$$

Note: β_i is the coefficient of regressing R_i on R_m using OLS.

This is the famous **CAPM** asset pricing equation.

Beta

- The risk premium for asset i depends on:
  - it's **beta** (essentially the correlation with the market)
  - the market price of risk: $E\ R_m - R$.
- A stock's beta can be estimated from data on past returns of the stock (R_i) and the market (using a broad stock index).
- Betas are used to
  - Measure the risk of an asset.
  - Calculate the required rate of return for investment projects.
  - Evaluation of mutual fund managers.

Securities market line

CAPM prediction:

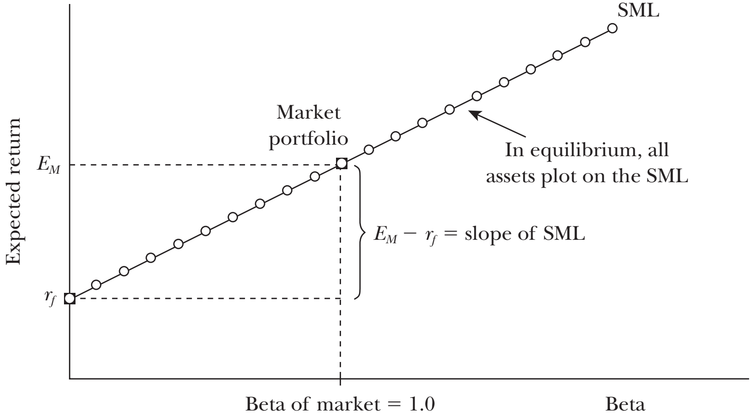
$$\mathbb{E}R_i = (1-\beta_i)R + \beta_i \mathbb{E}R_m \tag{22}$$

$$= R + \beta_i \mathbb{E}\{R_m - R\} \tag{23}$$

If we plot expected returns against βs, we should get a straight line.
This is called the **securities market line** (SML)

Securities market line

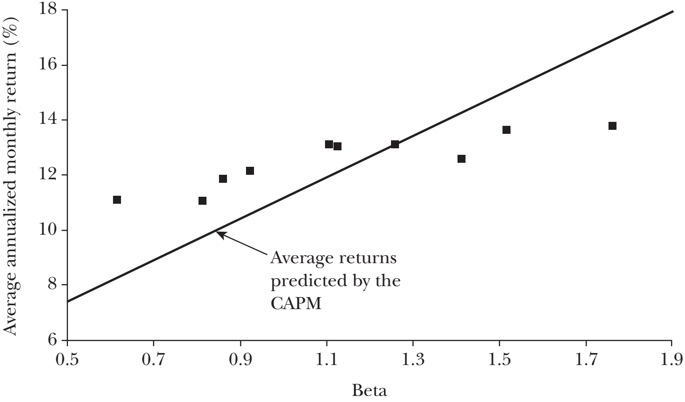
The Securities Market Line (SML)



Source: Perold (2004)

Securities market line: Evidence

Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003



Source: Fama (2004)

Implications

Stocks with higher βs have higher expected returns, but the relationship is flatter than predicted.

Again: we don't understand how investors value / measure risk.

- a fundamental problem.

Oddly, β remains popular, even though it does not work in the data.

5. Solving for the Asset Price

Solving for the asset price

We show that the asset price equals the present discounted value of dividends

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} d_{t+j} MRS(t, t+j) \tag{24}$$

The discount factor is the MRS, called the **stochastic discount factor**.

Solving for the asset price

Start from the Euler equation:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \tag{25}$$

Solve for the price:

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \tag{26}$$

Replace p_{t+1} with (26) shifted to $t+1$:

$$\begin{aligned} p_t &= E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\ &\quad E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[\frac{\beta u'(c_{t+2})}{u'(c_{t+1})} \right] (p_{t+2} + d_{t+2}) \right\} \end{aligned} \tag{27}$$

Solving for the asset price

The law of iterated expectations:

$$E_t \{E_{t+1}(x)\} = E_t(x) \tag{28}$$

Eliminate the E_{t+1}:

$$p_t = E_t \left\{ \frac{\beta\, u'(c_{t+1})}{u'(c_t)}\, d_{t+1} \right\} + E_t \left\{ \frac{\beta^2\, u'(c_{t+2})}{u'(c_t)}\, (p_{t+2} + d_{t+2}) \right\} \tag{29}$$

Iterate forward for T periods:

$$p_t = E_t \left\{ \sum_{j=1}^{T} \frac{\beta^j\, u'(c_{t+j})}{u'(c_t)}\, d_{t+j} \right\} \tag{30}$$

$$+E_t \left\{ \frac{\beta^T u'(c_{t+T})}{u'(c_t)}\, p_{t+T} \right\} \tag{31}$$

Solving for the asset price

Impose that the last term vanishes in the limit:

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \tag{32}$$

- There is no good reason for this assumption!
- We will see later: other prices solve the asset pricing equation (bubbles)

The asset price equals the **discounted present value of dividends**.

The stochastic **discount factor** is the marginal rate of substitution.

Example: Log Utility

In the Lucas model, assume: $u(c) = \ln(c)$. $K = 1$.

In equilibrium: $c_t = d_t$.

$MRS_{t+1} = \frac{\beta \ u'(c_{t+1})}{u'(c_t)} = \frac{\beta \ d_t}{d_{t+1}}$.

The asset pricing equation becomes

$$
\begin{aligned}
p_t &= E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j d_t}{d_{t+j}} d_{t+j} \right\} \\
&= d_t \frac{\beta}{1-\beta}
\end{aligned}
$$

Example: Periodic dividends

In the Lucas model, assume:

- Utility is $u(c) = c^{1-\sigma}/(1-\sigma)$.
- d_t alternates between d^H and d^L.

Asset pricing equation:

$$\begin{aligned} p_t &= \sum \beta^j (d_t/d_{t+j})^\sigma d_{t+j} \\ &= d_t^\sigma \sum \beta^j d_{t+j}^{1-\sigma} \end{aligned} \tag{33}$$

On good days, p_t is pulled up by low $u'(c')$, but is pushed down by low d_{t+1}.

5.2. The Excess Volatility Puzzle

Consider a stock with dividend process d_t.

Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \tag{34}$$

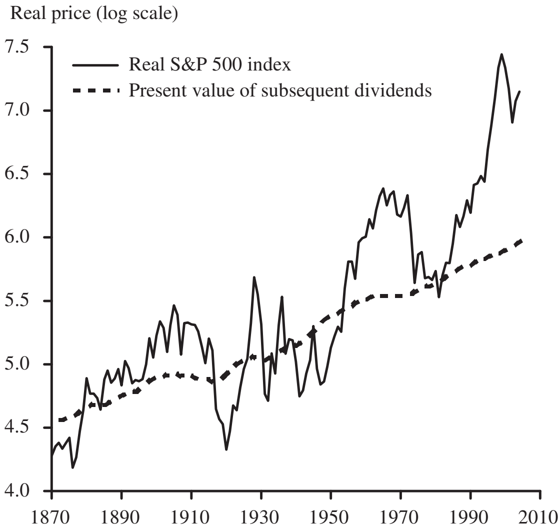
In the data:

- Dividends are very smooth (a goal of company policy).
- Stock prices are much more volatile than dividends.

But in the theory: stock prices should be the **average** of future dividends and thus **smoother** than dividends.

This is the flip-side of the Equity Premium Puzzle. See Shiller (1981)

Excess Volatility



Source: FRSBSF Economic Letter Nov 2007

5.3. Bubbles

- Recall how the asset pricing formula is derived:
- We iterate forward on the asset pricing Euler equation

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \tag{35}$$

- We assume that the p_{t+1} term vanishes in the limit.
- What if it does not vanish?
- Then **any** (current) **asset price** can satisfy the asset pricing equation.
- The deviation between p_t and the **fundamental price** from (35) is called a **bubble**.
- It is purely a self-fulfilling expectation.

Bubbles: Example

- Consider an asset that pays no dividends.
- Its **fundamental price** is 0.
- Assume that the MRS is constant at $\frac{\beta \ u'(c_{t+1})}{u'(c_1)} = 1$.
- The the asset pricing equation is

$$p_t = E_t p_{t+1} \tag{36}$$

- One price process that satisfies this: p doubles with probability 1/2 and drops to 0 otherwise.
- This satisfies (36) for **any** p_t.
- Bubbles are a possible explanation for asset price volatility.
- Note that the bubble does not offer any excess return opportunities.

Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.
- On the equity premium puzzle: Mehra and Prescott (1985, 2003)

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