

# Overlapping Generations Model

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# Introduction

Two approaches for modeling the household sector

1. households live forever (**infinite horizon**)  
tractable
2. households live for finite number of periods (**overlapping generations**)  
can talk about questions where demographics matter

# OLG Applications

Fiscal policy analysis

- ▶ often models where households live many periods

Wealth and income inequality

Economic growth

Many others...

## What we do in this section

How to set up and solve an OLG model

Show that the world is **not efficient**: households may save too much.

“Social security” can prevent overaccumulation

We can make households "infinitely lived" by adding altruistic **bequests**.

## What we don't do in this section

- ▶ We sidestep some technical issues:
  - ▶ why is there a representative household?
  - ▶ why is there a representative firm?
- ▶ We come back to those later.

# An OLG Model Without Firms

# Steps

We go through the standard steps:

1. Describe the economy: demographics, endowments, preferences, technologies, markets
2. Solve each agent's problem
3. Market clearing
4. Competitive equilibrium

# Demographics

- ▶ At each date a cohort of size

$$N_t = N_0(1 + n)^t$$

is born.

- ▶ Each person lives for two periods.
- ▶ Therefore, at each date there are  $N_t$  young and  $N_{t-1}$  old households.



# Endowments, Preferences

- ▶ Endowments
  - ▶ Young households receive endowments  $w_t$ .
- ▶ Preferences:  $u(c_t^y) + \beta u(c_{t+1}^o)$ .

# Technology

- ▶ Endowments can be stored.
- ▶ Storing  $s_t$  today yields  $f(s_t)$  tomorrow.
- ▶ Resource constraint:

$$N_t c_t^y + N_{t-1} c_t^o + N_t s_t = N_t w_t + N_{t-1} f(s_{t-1}) \quad (1)$$

## Resource constraints

Technological constraints that describe the set of feasible choices.  
Often identical to market clearing conditions.

# Markets

Goods are traded in spot markets.

Households can issue one period bonds with interest rate  $r_{t+1}$ .

We are done with the description of the environment.

Next step: solve the household problem.

## A Missing Market

Even though there is a bond market, **intergenerational** borrowing and lending is not possible.

The reason: the young at  $t$  cannot borrow from the old because the old won't be around at  $t+1$  to have their loans repaid.

- ▶ If households live for more periods, the problem becomes weaker, but does not go away.

An asset that stays around forever solves this problem

- ▶ e.g., money, land, shares

# Household Problem

- ▶ The budget constraints are

$$\begin{aligned}w_t &= c_t^y + s_{t+1} + b_{t+1} \\ c_{t+1}^o &= f(s_{t+1}) + b_{t+1}(1 + r_{t+1})\end{aligned}$$

- ▶ Lifetime budget constraint:

$$w_t - c_t^y - s_{t+1} = [c_{t+1}^o - f(s_{t+1})]/[1 + r_{t+1}]$$

# Lagrangian

$$\Gamma = u(c_t^y) + \beta u(c_{t+1}^o) + \lambda_t \{ [w_t - c_t^y - s_{t+1}] - [c_{t+1}^o - f(s_{t+1})] / [1 + r_{t+1}] \}$$

FOCs:

$$\begin{aligned} u'(c_t^y) &= \lambda_t \\ \beta u'(c_{t+1}^o) &= \lambda_t / (1 + r_{t+1}) \\ f'(s_{t+1}) &= 1 + r_{t+1} \end{aligned}$$

## Euler equation

$$u'(c_t^y) = \beta (1 + r_{t+1}) u'(c_{t+1}^o)$$

### Interpretation:

Give up 1 unit of consumption when young and buy a bond.

Marginal cost:  $u'(c_t^y)$

Marginal benefit:

$(1 + r_{t+1})$  units of consumption when old

valued at  $\beta u'(c_{t+1}^o)$

# Household Solution

A vector  $(c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})$  which satisfies

- ▶ 2 FOCs (an EE and the foc for  $s$ )
- ▶ 2 budget constraints.



# Equilibrium

A CE is an allocation  $\{c_t^y, c_t^o, s_t, b_t\}$  and a price system  $\{r_t\}$  that satisfy:

- ▶ 4 household conditions
- ▶ bond market clearing:  $b_t = 0$ ;
- ▶ goods market clearing (same as resource constraint):

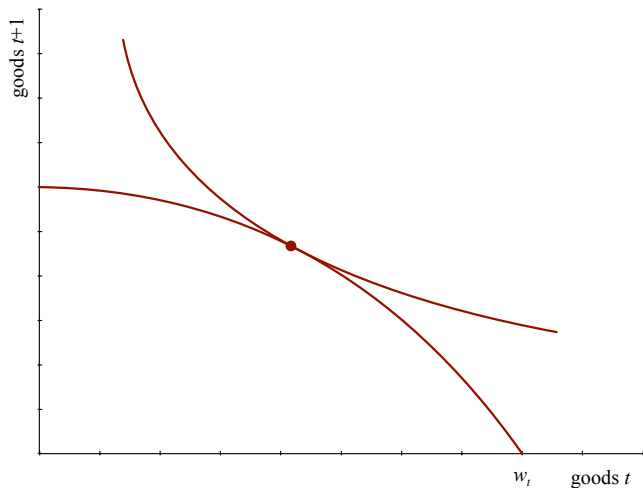
$$N_t c_t^y + N_{t-1} c_t^o + N_t s_t = N_t w_t + N_{t-1} f(s_{t-1}) \quad (2)$$

We are done with the definition of equilibrium.

Next step: characterize equilibrium.

# Characterization

There is no trade in equilibrium ( $b_t = 0$ )



# A Production Economy

# A Production Economy

- ▶ The model is modified by adding firms who rent capital and labor from households.
- ▶ The endowment  $w$  is now interpreted as labor earnings.
- ▶ Households supply one unit of labor inelastically to firms when young.
- ▶ Capital depreciates at rate  $\delta$ .

# Model Elements

- ▶ Unchanged: demographics, preferences
- ▶ Endowments:
  - ▶ at  $t = 0$  each old household owns  $k_0$  units of capital
  - ▶ each young has 1 unit of work time
- ▶ Technology

$$F(K_t, L_t) + (1 - \delta)K_t = C_t + K_{t+1} \quad (3)$$

- ▶ constant returns to scale
  - ▶ Inada conditions
- ▶ Markets:
  - ▶ goods (numeraire), capital rental ( $q$ ), labor rental ( $w$ )

# Notes

## Representative household

- ▶ All households are the same.
- ▶ So we talk as if there were only 1 household, who behaves competitively.

## The household owns everything

- ▶ The firm rents capital from the household in each period
- ▶ That makes the firms' problem static (easy)
- ▶ It is usually convenient to pack all dynamic decisions into 1 agent
- ▶ In this model, who owns the capital makes no difference - why not?

# Households

- ▶ Budget constraints:

$$\begin{aligned}w_t &= c_t^y + s_{t+1} + b_{t+1} \\c_{t+1}^o &= e^o + (s_{t+1} + b_{t+1})(1 + r_{t+1})\end{aligned}$$

- ▶  $e^o$ : any other income received when old (currently 0)
- ▶ There are no profits b/c the technology has constant returns to scale.

## Lifetime budget constraint

- ▶ Combine the 2 budget constraints:

$$w_t - c_t^y = (c_{t+1}^o - e^o) / [1 + r_{t+1}]$$

or

$$W_t = w_t + \frac{e^o}{1 + r_{t+1}} = c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} \quad (4)$$

- ▶  $W_t$ : present value of lifetime earnings



# Permanent Income Hypothesis

The lifetime budget constraint only depends on  $W_t$ , not on timing of income over life.

Therefore, the optimal consumption path only depends on  $W_t$ .

This is a somewhat general implication that has been tested many times.

A recent example:

- ▶ Hsieh (2003) [Nice example of using a natural experiment to test a theory.]

Overall, the evidence seems favorable.

# Lagrangian

$$\Gamma = u(c_t^y) + \beta u(c_{t+1}^o) + \lambda \{W_t - c_t^y - c_{t+1}^o / [1 + r_{t+1}]\}$$

FOCs:

$$\begin{aligned} u'(c_t^y) &= \lambda \\ \beta u'(c_{t+1}^o) &= \lambda / (1 + r_{t+1}) \end{aligned}$$

# Households

Euler:

$$u'(c_t^y) = \beta(1 + r_{t+1})u'(c_{t+1}^o)$$

Solution: A vector  $(c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})$  that satisfies 2 budget constraints and 1 EE.

We lack one equation! Why?

## Intuition: Household Solution

The interest rate determines the **slope** of the age-consumption profile.

Lifetime income determines the **level**.

This is especially clear with log utility:  $c_{t+1}^o/c_t^y = \beta(1+r_{t+1})$

Graph...

## Firms

Firms maximize current period profits taking factor prices  $(q, w)$  as given.

$$\max F(K, L) - wL - qK$$

FOCs:

$$q = F_K(K, L)$$

$$w = F_L(K, L)$$

The **solution** to the firm's problem is a pair  $(K, L)$  so that the 2 FOCs hold.

A wrinkle: We assume constant returns to scale.

- ▶ the size of the firm is indeterminate
- ▶ the FOCs determine  $K/L$  as a function of  $q/w$ .

## Firms: Intensive form

It is convenient to write the production function in **intensive form**:

$$\begin{aligned} F(K,L) &= LF(K/L,1) \\ &= Lf(k^F) \end{aligned}$$

where  $k^F = K/L$  and

$$f(k^F) = F(k^F, 1)$$

## Firms: Intensive form

Now the factor prices are

$$F_K = Lf'(k^F)(1/L)$$

and

$$\begin{aligned} F_L &= f(k^F) + Lf'(k^F)(-K/L^2) \\ &= f(k^F) - f'(k^F)k^F \end{aligned}$$

Therefore:

$$\begin{aligned} q &= f'(k^F) \\ w &= f(k^F) - k^F f'(k^F) \end{aligned}$$

Important:  $q$  is the rental price of capital, which differs from the interest rate  $r$ .

# Market clearing

Capital rental:

Labor rental:

Bonds:

Goods:



# Competitive Equilibrium

An allocation:

Prices:

That satisfy:

We have 9 objects and 9 equations – one is missing.

We need an accounting identity linking  $r$  and  $q$ :

- ▶ The household receives  $1 + r_{t+1} = q_{t+1} + 1 - \delta$  per unit of capital.
- ▶ Therefore,  $r = q - \delta$ .

## Reading

- ▶ Acemoglu (2009), ch. 9.
- ▶ Krueger, "Macroeconomic Theory," ch. 8
- ▶ Ljungqvist and Sargent (2004), ch. 9 (without the monetary parts).
- ▶ McCandless and Wallace (1991) and De La Croix and Michel (2002) are book-length treatments of overlapping generations models.

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- Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.
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