

The Growth Model In Continuous Time: Solow Model

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Topics

- ▶ We study the standard growth model in continuous time.
- ▶ To solve it: Optimal Control
- ▶ To characterize it: phase diagrams

Continuous Time vs. Discrete Time

[Some of you will find the next several slides obvious.]

Continuous time

- ▶ So far, time was divided into discrete "periods."
- ▶ It is often more convenient to shrink the length of periods to 0.
- ▶ Difference equations then become differential equations.

Continuous time

Example: Law of motion for capital

- ▶ Discrete time:

$$K_{t+1} - K_t = I_t - \delta K_t \quad (1)$$

- ▶ More generally:

$$K_{t+\Delta t} - K_t = [I_t - \delta K_t] \Delta t \quad (2)$$

- ▶ Continuous time ($\Delta t \rightarrow 0$):

$$\lim_{\Delta t \rightarrow 0} \frac{K_{t+\Delta t} - K_t}{\Delta t} = \dot{K}_t = I_t - \delta K_t \quad (3)$$

Notation: $\dot{K} = dK/dt$.

Growth rates in continuous time

The growth rate of a variable is defined as

$$g(x) = \frac{\dot{x}}{x} = \frac{d \ln x}{dt} \quad (4)$$

Growth rate rules (easy to prove):

1. $g(xy) = g(x) + g(y)$.
2. $g(x/y) = g(x) - g(y)$.
3. $g(x^\alpha) = \alpha g(x)$.
4. $x(t) = e^{\gamma t} \implies g(x) = \gamma$.

Differential equations

Differential equations

Take a function of time:

$$x(t) = a + bt \quad (5)$$

There is another way of describing this function:

- ▶ Take the derivative:

$$\dot{x}(t) = dx(t)/dt = b \quad (6)$$

- ▶ Fix $x(0) = a$.
- ▶ The two pieces of information (the derivative and $x(0)$) completely describe $x(t)$.
- ▶ Only one function $x(t)$ satisfies both pieces.
 - ▶ But note that infinitely many functions satisfy the derivative!

Definition: Differential equation

- ▶ A differential equation (DE) is a function of the form

$$\dot{x}(t) = f(x(t), t) \quad (7)$$

- ▶ This is actually a "first-order" DE.
- ▶ **Higher order** DEs contain higher order derivatives of time.
 - ▶ E.g.: A second order DE

$$d^2x(t)/dt^2 + dx(t)/dt = a + bt \quad (8)$$

- ▶ Together with a boundary condition, the DE can be solved for $x(t)$.

Solving DEs

- ▶ The bad news: There is no algorithm for solving DEs.
- ▶ But one look up solutions in tables.
- ▶ It is also easy to **verify** a solution one may guess.

Guess + Verify

Consider again

$$\dot{x}(t) = b \quad (9)$$

$$x(0) = a \quad (10)$$

Guess

$$x(t) = a + bt \quad (11)$$

Verify:

- ▶ Take the time derivative and find that it matches $\dot{x} = b$.
- ▶ Verify that $x(0) = a$.

Example: constant growth

$$\dot{x}(t) = b x(t) \quad (12)$$

$$x(0) = a \quad (13)$$

Guess:

$$x(t) = a e^{bt} \quad (14)$$

Verify: Take the derivative

$$\dot{x}(t) = b a e^{bt} = b x(t) \quad (15)$$

$$x(0) = a e^0 = a \quad (16)$$

Boundary conditions

Boundary conditions can take many forms:

- ▶ $\int_a^b x(s) ds = 5.$
- ▶ $x(1.7) = 5.$
- ▶ $x(T) - x(T - 2) = 5.$
- ▶ etc.

The Solow Model

The Solow Model - Structure

- ▶ Modify the discrete time growth model in two ways:
 1. Continuous time.
 2. Fixed saving rate.
- ▶ This is not an equilibrium model, but can be interpreted as one.

Model Elements

- ▶ Demographics: households live forever;

$$L_t = e^{nt} \quad (17)$$

- ▶ Preferences:

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (18)$$

- ▶ Endowments:

- ▶ at each moment, the household has 1 unit of work time
- ▶ at $t = 0$ he has K_0 goods

Model Elements

Technology:

$$F(K_t, L_t) = \dot{K}_t + \delta K_t + L_t c_t \quad (19)$$

- ▶ F : constant returns to scale

Markets:

- ▶ competitive markets for goods (numeraire), labor rental, capital rental

Firms

The firm solves a static problem.

The same as in discrete time.

$$\max F(K, L) - wL - qK \quad (20)$$

FOC

$$q_t = F_K \quad (21)$$

$$w_t = F_L \quad (22)$$

Firms: Intensive Form

Define $k^F = K/L$ and

$$f(k^F) = F(K, L) / L = F(k^F, 1) \quad (23)$$

The first order conditions are then

$$q = f'(k^F) \quad (24)$$

and

$$w = f(k^F) - f'(k^F)k^F \quad (25)$$

Households

Budget constraint

$$\dot{K}_t = w_t L_t + (q_t - \delta) K_t - L_t c_t$$

It is convenient to have everything per capita.

Define $k = K/L$.

Law of motion for k :

$$\begin{aligned}\dot{k}/k &= \dot{K}/K - n \\ &= w/k + (q - \delta) - c/k\end{aligned}$$

Or

$$\dot{k}_t = w_t + (q_t - \delta - n)k_t - c_t \quad (26)$$

Constant saving rate

- ▶ The modern way: Set up an optimization problem and derive the saving function.
- ▶ The Solow way: Assume that the saving rate is fixed:

$$c = (1 - s)(w + qk) \quad (27)$$

- ▶ Therefore:

$$\dot{k} = s(w + qk) - (n + \delta)k \quad (28)$$

Market Clearing

Capital rental:

$$k = k^F \quad (29)$$

Goods market:

$$F(K_t, L_t) = C_t + \delta K_t + \dot{K}_t$$

or in per capita terms

$$\dot{k} = f(k) - (n + \delta)k - c \quad (30)$$

Equilibrium

An equilibrium is a collection of *functions* (of time)

$$c_t, k_t, k_t^F, w_t, q_t$$

that satisfy

1. the firm's first order conditions (2)
2. the household's budget constraint and the behavioral equation

$$\dot{k} = s(w + qk) - (n + \delta)k$$

3. market clearing (2)

Law of Motion

- ▶ The entire model boils down to to one key equation:

$$\dot{k}_t = sf(k_t) - (n + \delta)k_t \quad (31)$$

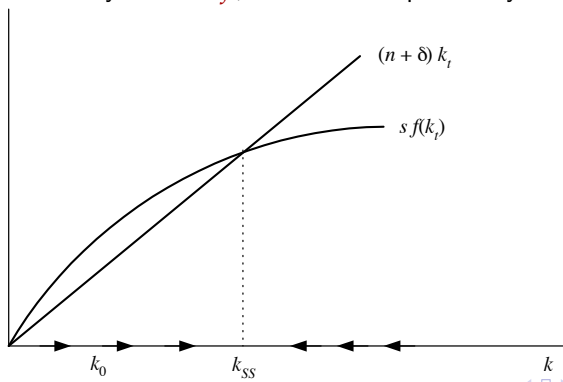
- ▶ This is simply the household's behavioral equation after applying $f(k) = w + qk$.

Steady state

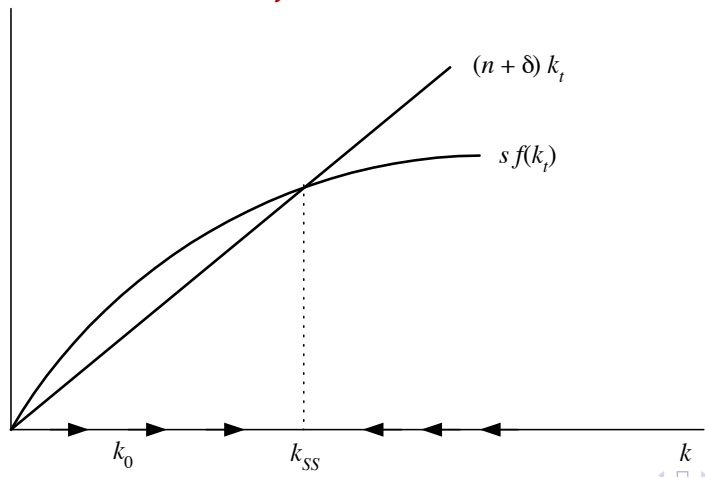
The steady state requires $\dot{k} = 0$ or

$$sf(k) = (n + \delta)k \quad (32)$$

With strictly concave f , there is a unique steady state with $k > 0$.



Dynamics



The steady state is *stable*.

Convergence is monotone.

Adding Technical Change

- ▶ The model does not have sustained growth in per capita income.
- ▶ This requires technical change (A grows).
- ▶ Assume exogenous growth in A :

$$A(t) = A(0)e^{\gamma t} \quad (33)$$

Adding Technical Change

- ▶ Assume that technical change takes the following form:

$$Y(t) = F(K(t), A(t)L(t)) \quad (34)$$

- ▶ This type of technical change is called “labor-augmenting” or “Harrod-neutral.”
- ▶ This is the *only* form of technical change that is consistent with *balanced growth*.

Definition

A balanced growth path is a path along which all growth rates are constant.

How to analyze a growing model?

- ▶ Construct a **stationary transformation**.
- ▶ Divide each variable by its balanced growth factor:

$$\tilde{x}(t) = x(t)e^{-g_x t} \quad (35)$$

where g_x is the **balanced growth rate** of x .

- ▶ Or take ratios of variables that grow at the same rate.
- ▶ The economy in transformed variables (\tilde{x}) has a steady state.

How to find the balanced growth rates?

- ▶ For equations that involve sums:

$$Y(t) = C(t) + I(t) + G(t) \quad (36)$$

Constant growth (usually) requires that all summands grow at the same rate.

- ▶ For other equations: Try taking the growth rate of the whole equation.
- ▶ Example:

$$Y(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha} \quad (37)$$

implies

$$g(Y) = \alpha g(K) + (1 - \alpha)[g(A) + n] \quad (38)$$

Balanced growth path: Solow Model

- ▶ Start from

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t) \quad (39)$$

$$g(K(t)) = sF\left(1, \frac{A(t)L(t)}{K(t)}\right) - \delta \quad (40)$$

- ▶ Constant growth requires that

$$\bar{k}(t) = \frac{K(t)}{A(t)L(t)} \quad (41)$$

be constant over time. Thus, on a balanced growth path:

$$g(K) = \gamma + n \quad (42)$$

Balanced growth path

- ▶ Production function:

$$\bar{y}(t) = \frac{Y(t)}{A(t)L(t)} = F(\bar{k}(t), 1) \quad (43)$$

must be constant on a balanced growth path.

- ▶ Thus: The model has a steady state in (\bar{k}, \bar{y}) .

Law of motion

$$\begin{aligned}g(\bar{k}) &= g(K) - \gamma - n \\ &= sF\left(1, \frac{A(t)L(t)}{K(t)}\right) - \delta - \gamma - n \\ &= sf(\bar{k})/\bar{k} - \delta - \gamma - n\end{aligned}$$

Or

$$\dot{\bar{k}}(t) = sf(\bar{k}(t)) - (n + \delta + \gamma)\bar{k}(t) \quad (44)$$

Nothing changes, except the constant term in the law of motion.

Reading

- ▶ Acemoglu (2009), ch. 2 covers the Solow model and stationary transformations of growing economies.
- ▶ Barro and Martin (1995), ch. 1
- ▶ Romer (2011), ch. 1
- ▶ Krusell (2014) ch. 2 discusses some insights that might be gained from the Solow model (and its limitations).

References I

- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.
- Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.
- Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.